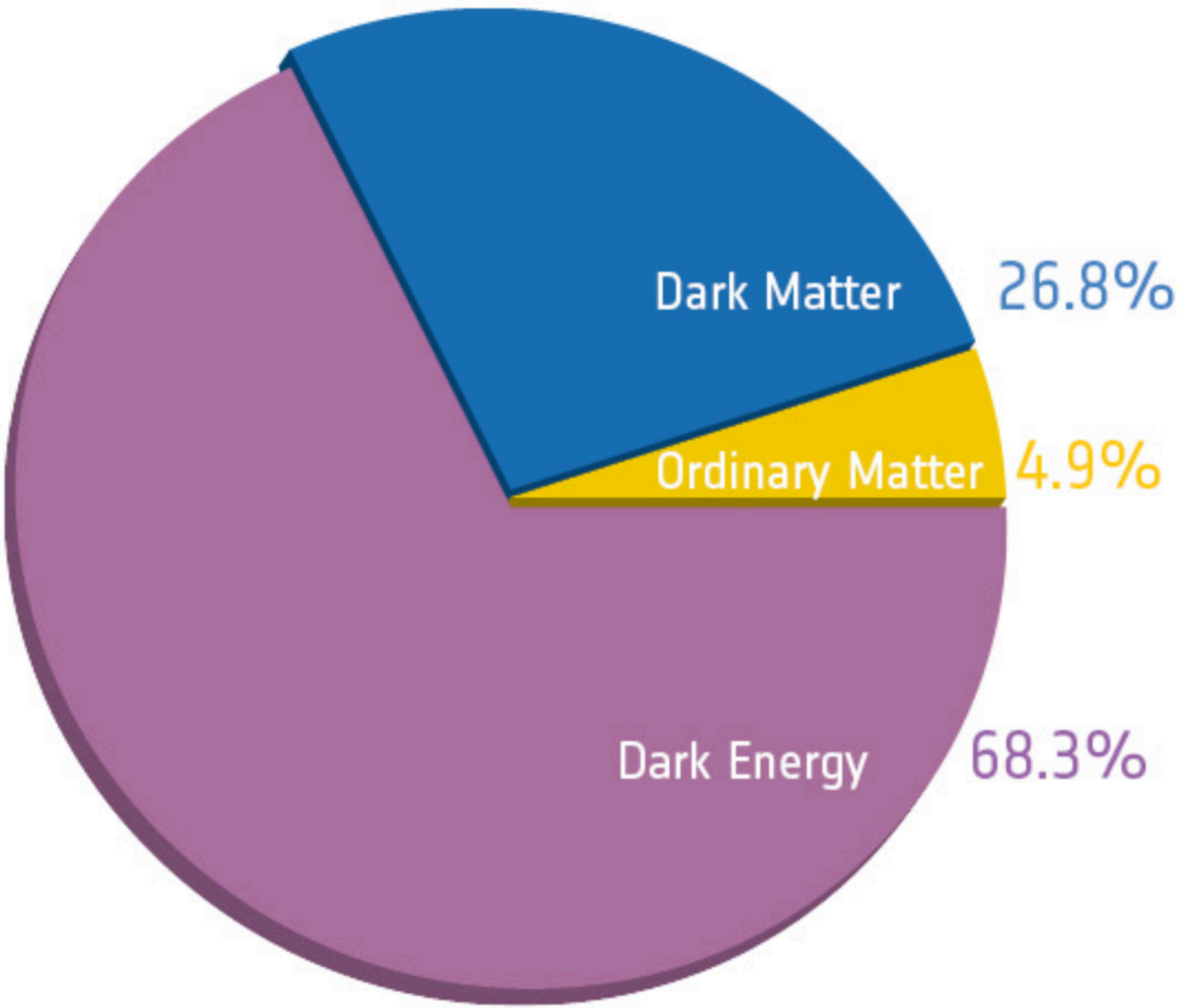
Imprint of the dark components on the CMB and LSS: part I

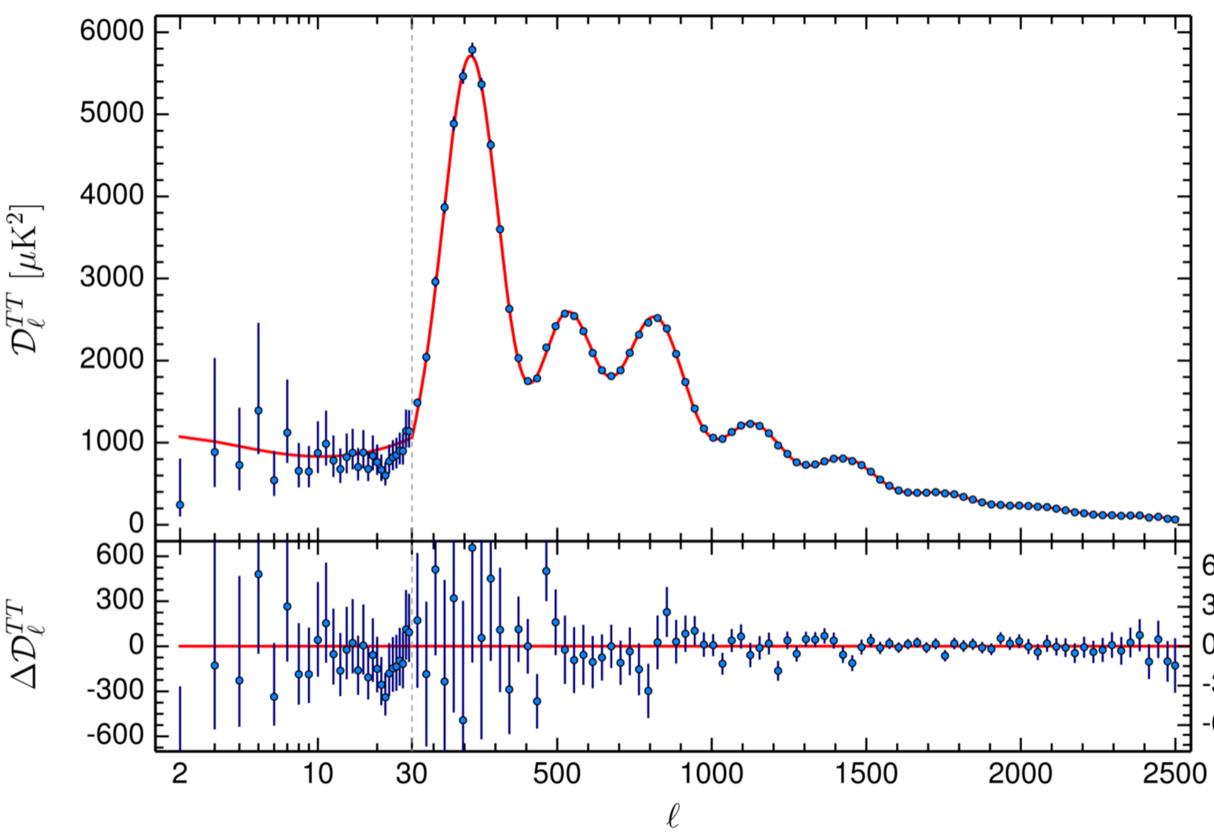
Thibaut Louis

Standard model of cosmology



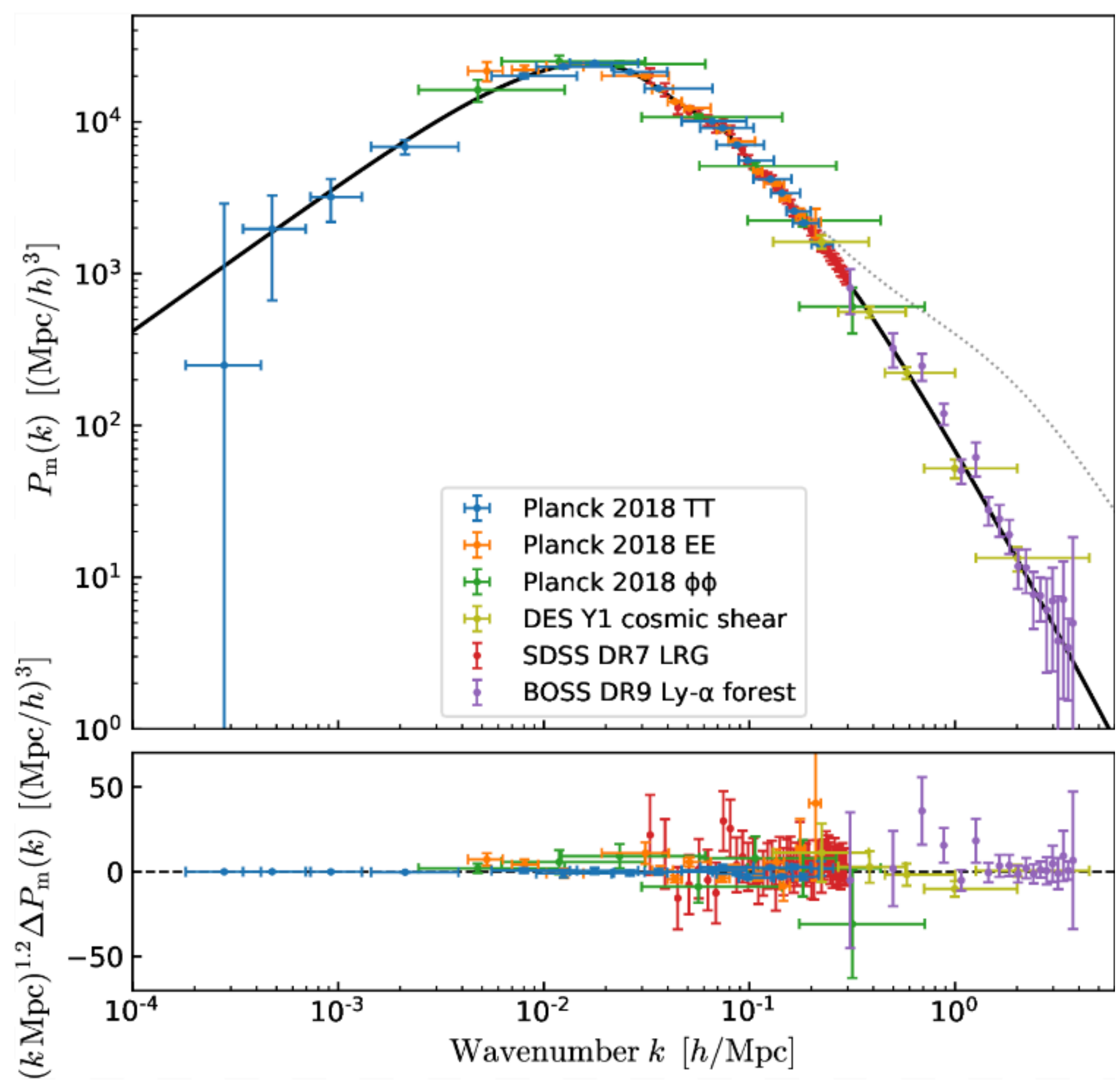
Standard model of cosmology

CMB anisotropy



https://arxiv.org/abs/1807.06205

Matter distribution



https://arxiv.org/pdf/1905.08103.pdf

Before we start: some non cosmological probes

Galaxy rotation curve

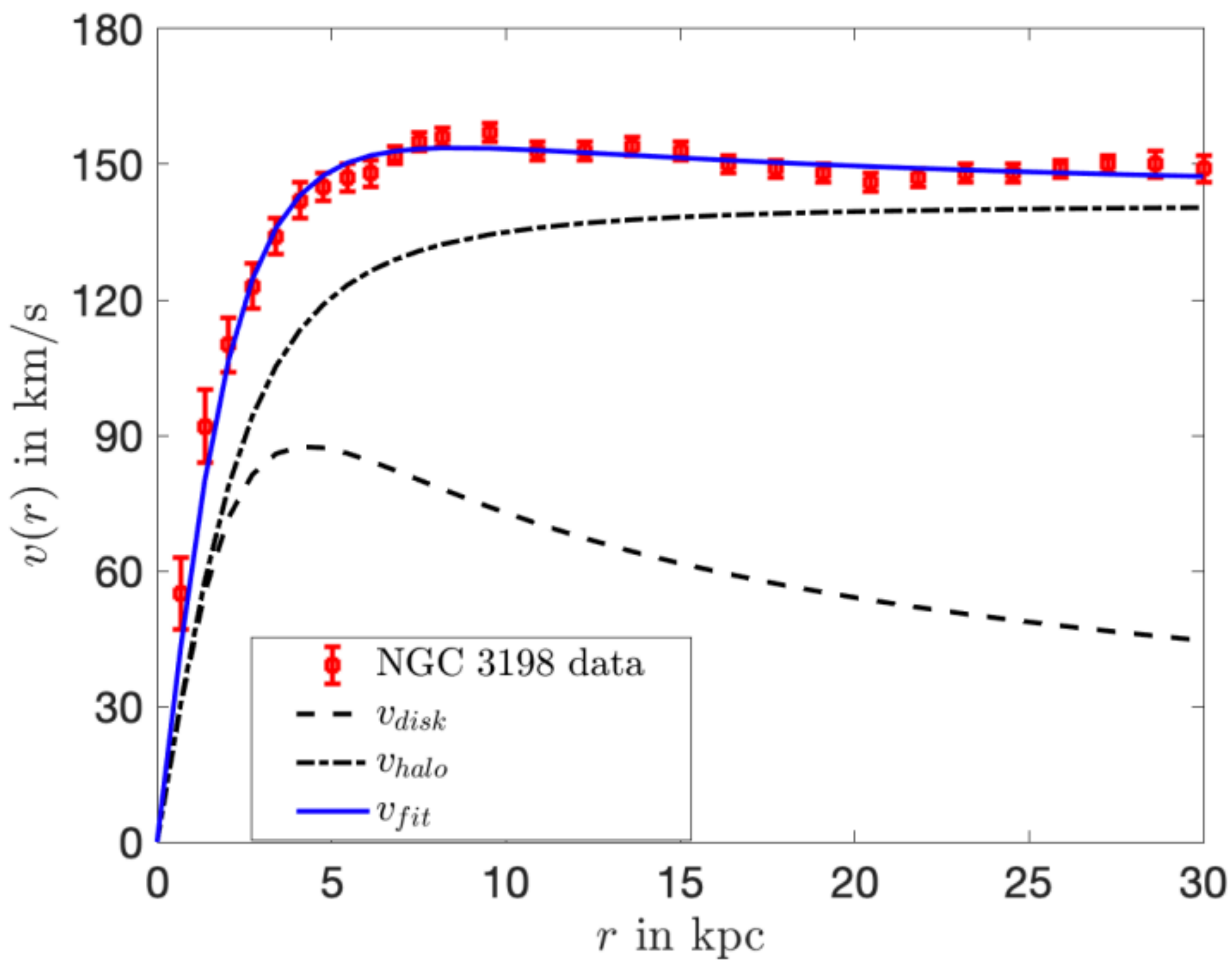
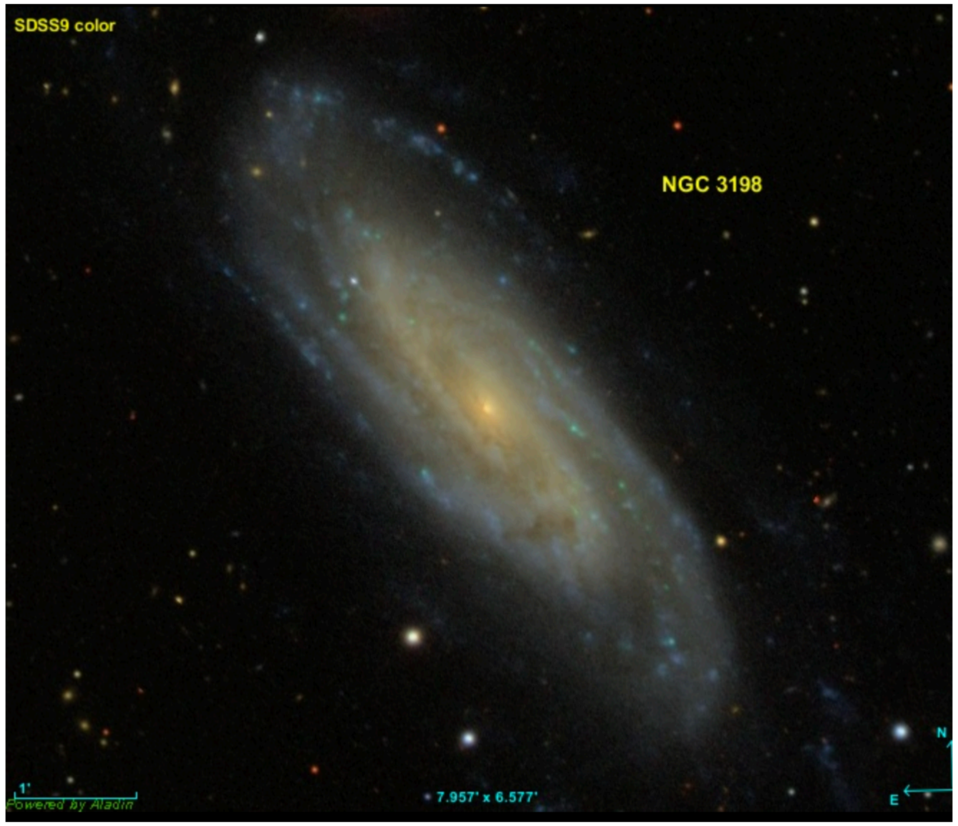


Fig. 1. The rotation curve for NGC 3198. r is the distance from the galactic center and $v(r)$ is the rotation speed. The data is from van Albada *et al* [6].



Dark matter really ? or modified gravity ?

https://arxiv.org/pdf/astro-ph/0207469.pdf

Example MOND:

$$\vec{F} = m.\,\mu\left(\frac{a}{a_0}\right).\vec{a}, \text{ avec } a = |\vec{a}| \quad \text{et}$$

$$\mu(x) = 1 \text{ si } x \gg 1$$
$$\mu(x) = x \text{ si } |x| \ll 1$$

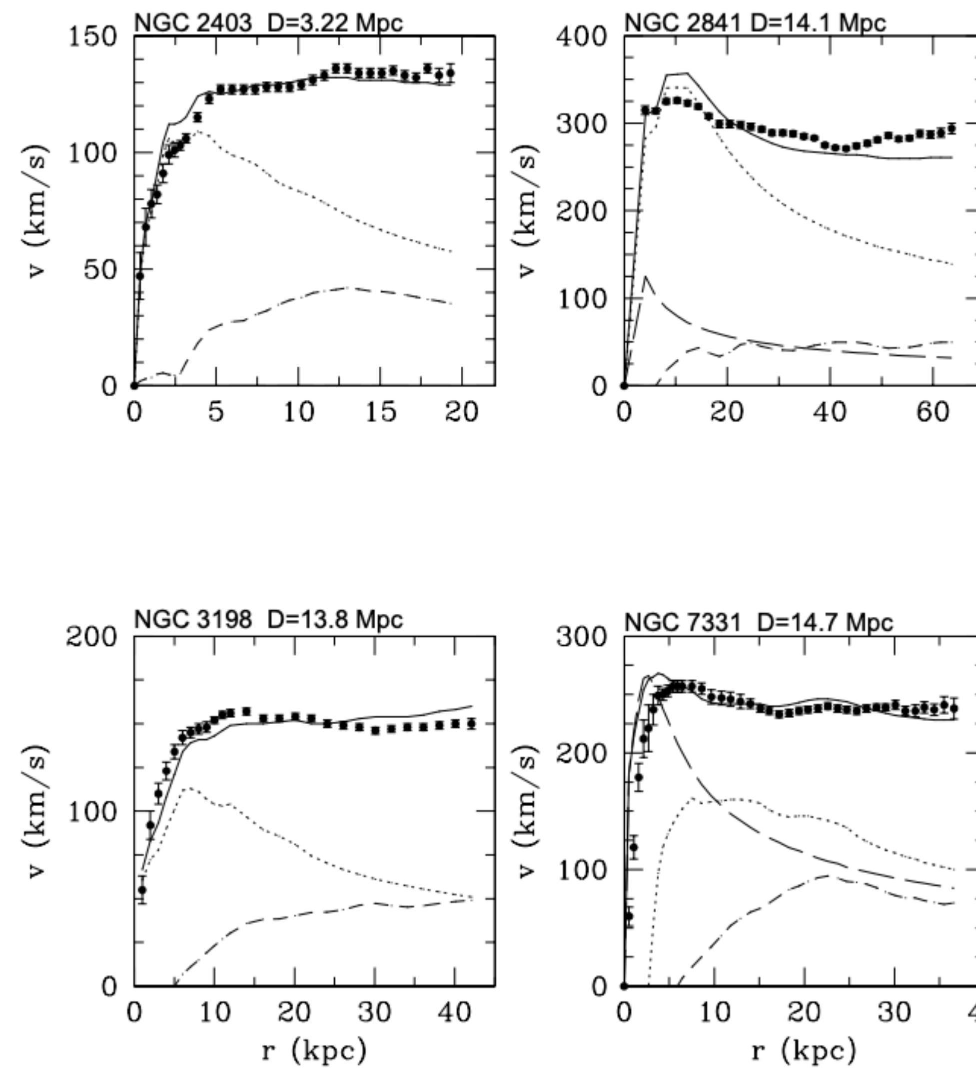
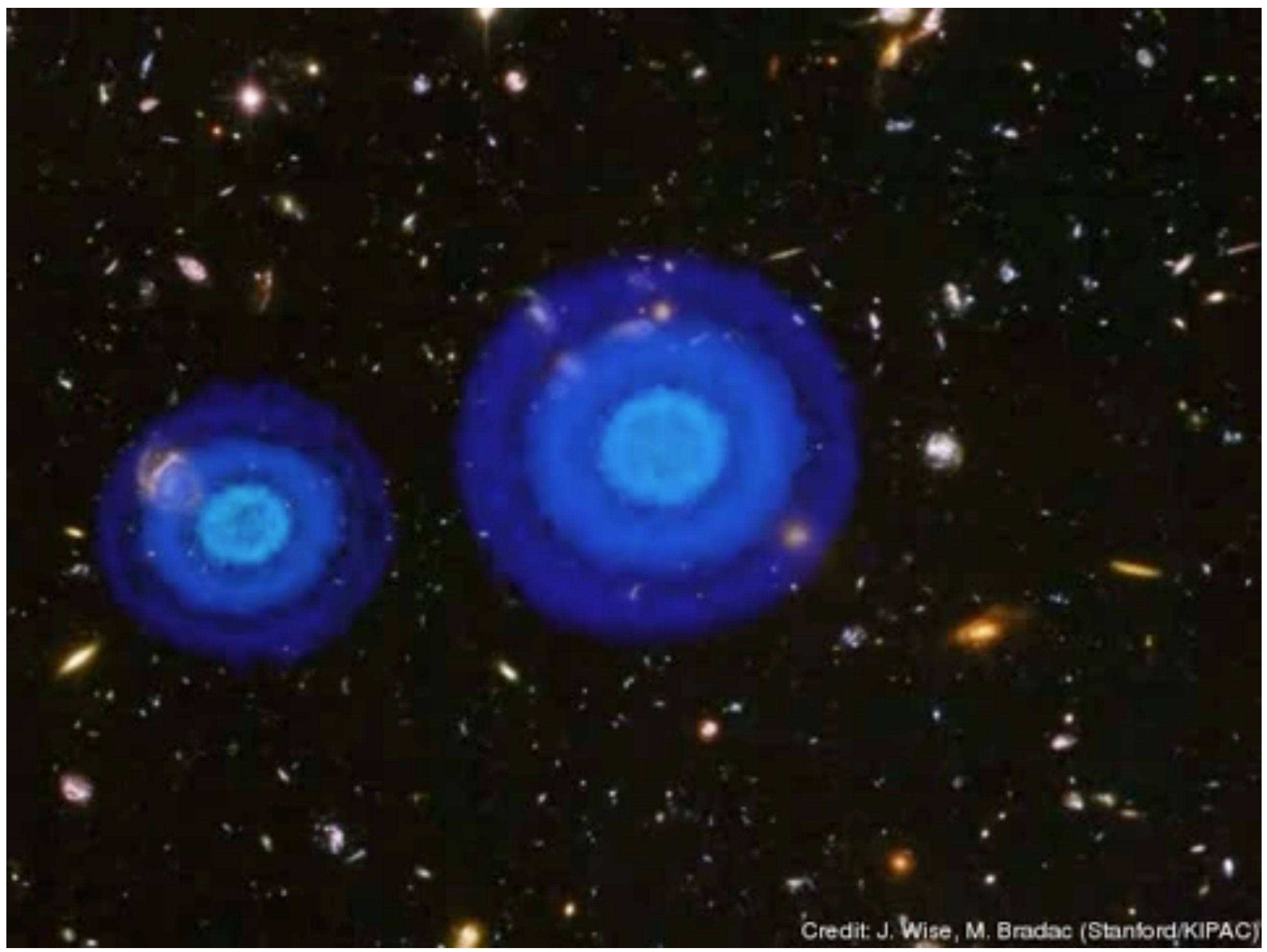


Fig. 1. MOND rotation curves compared to observed H I rotation curves for the four galaxies from the sample of BBS with Cepheid-based distances. The dotted, long-dashed, and short-dashed lines are the Newtonian rotation curves of the stellar disc, bulge, and gaseous components respectively.

Dark matter really ? or modified gravity ?

More challenging for modified gravity : the bullet cluster



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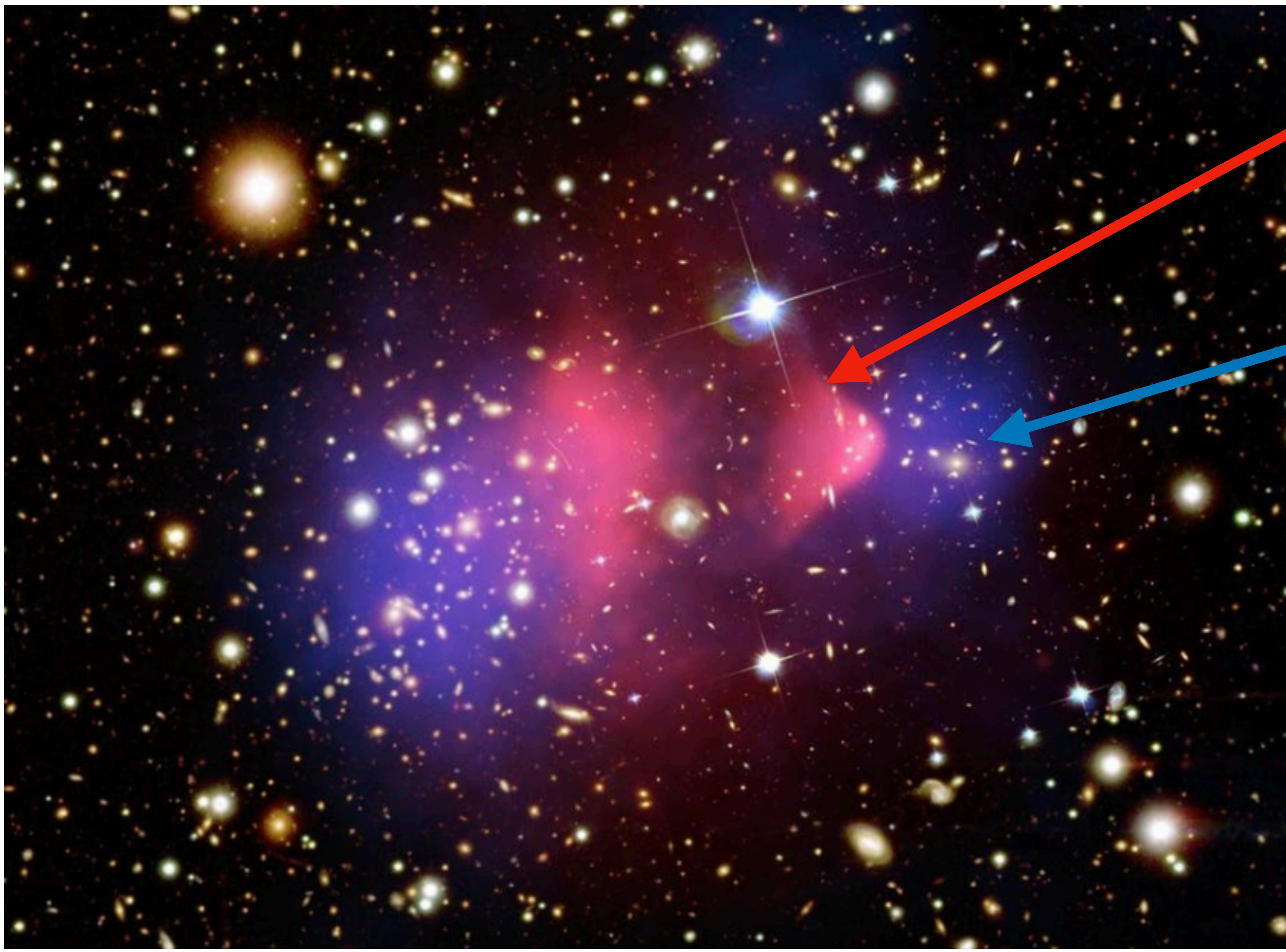


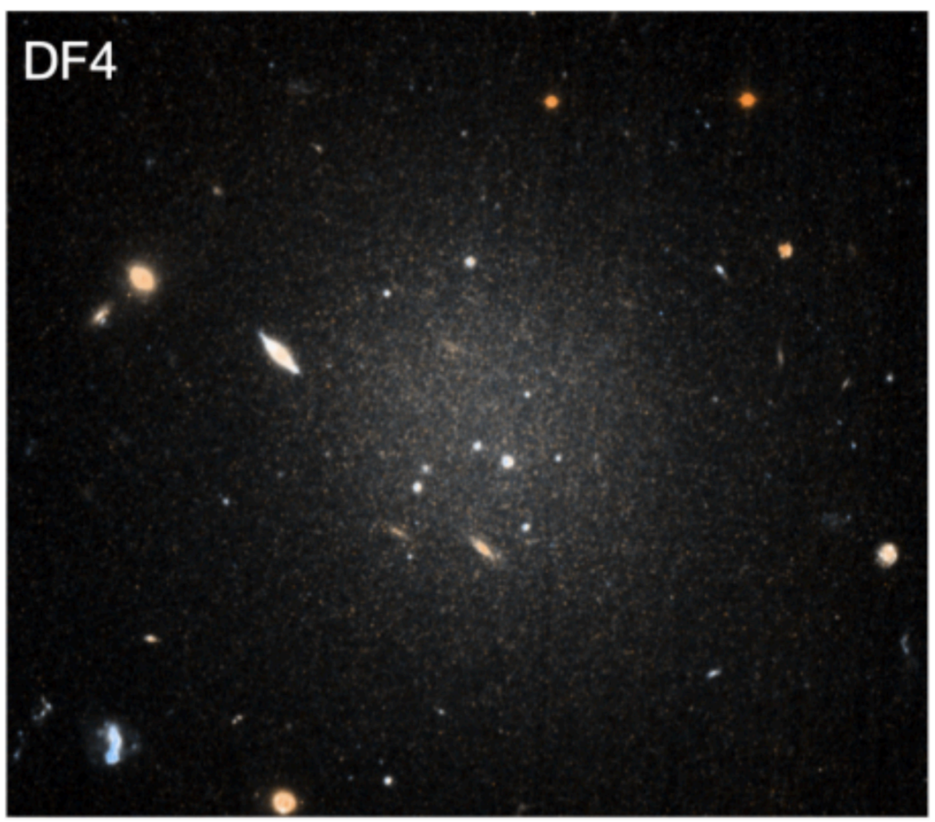
X-rays (gas)

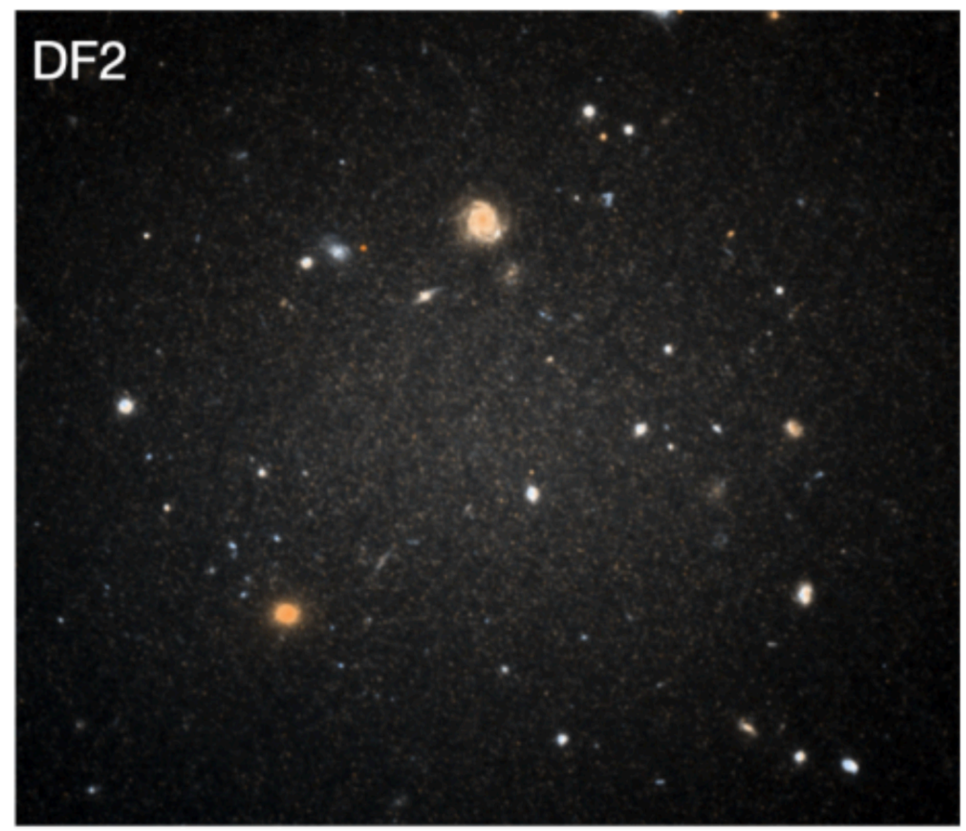
Lensing potential (matter)

Dark matter really ? or modified gravity ?

Some galaxies seems to have a deficit on dark matter

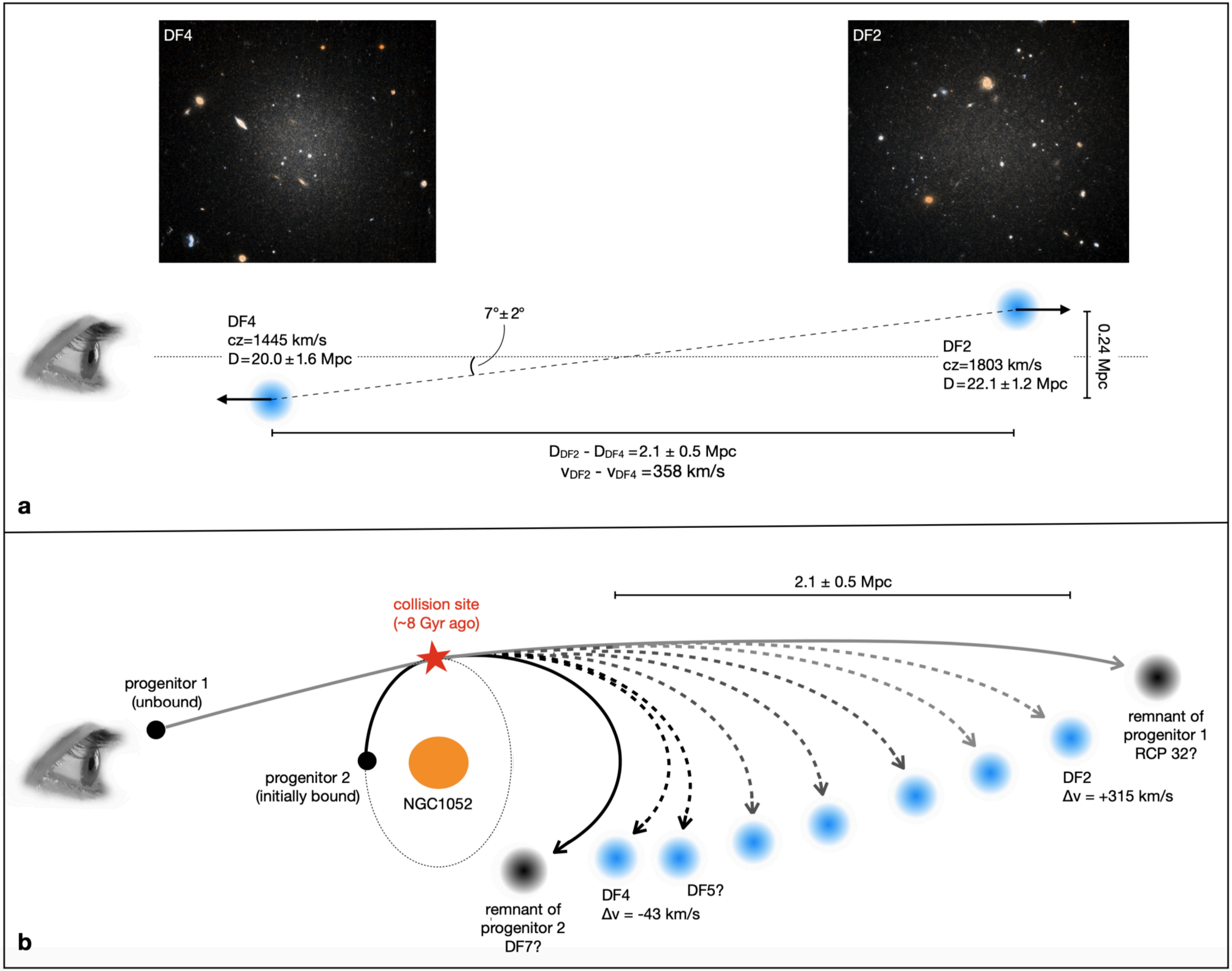
NGC 1052-DF2 and NGC 1052-DF4 are ultra diffuse galaxy whose kinematic can be explain without dark matter





e.g **https://www.nature.com/articles/s41586-022-04665-6**

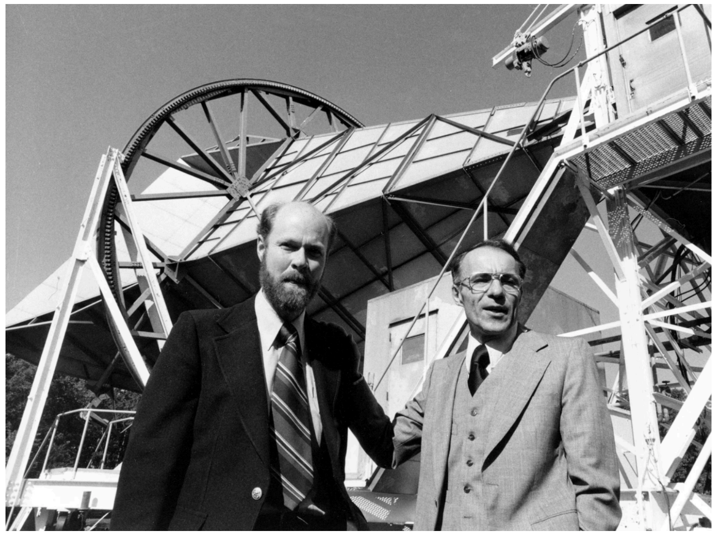
A possible scenario: (still controversial)



The fact that DM do not interact with baryonic matter is key to understand the CMB and LSS distribution, in the following we will discuss how exactly CMB physics depends on the matter content in the universe

The cosmic microwave background

First discovered in 1964 by Penzias and Wilson
Their measurement clearly showing the presence of the microwave background, with their instrument having an excess 4.2K antenna temperature which they could not account for.



https://articles.adsabs.harvard.edu/full/seri/ApJ../0142//0000418.000.html

Fun fact: Penzias phoned a friend at MIT, for unrelated reason.
Burke asked about the progress of the experiment, Burke had
Recently spoken with one of his colleague Ken Turner, who was just back from a visit
at Princeton, during which he had followed a seminar by Peebles about nucleosynthesis and possible relic radiation.

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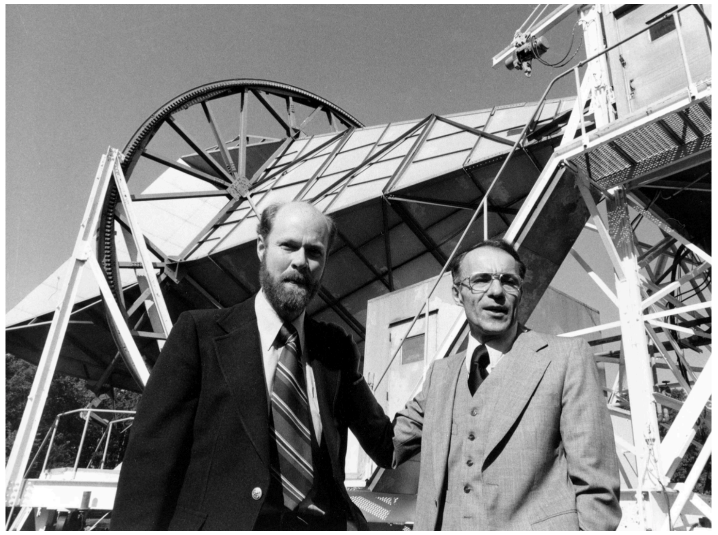

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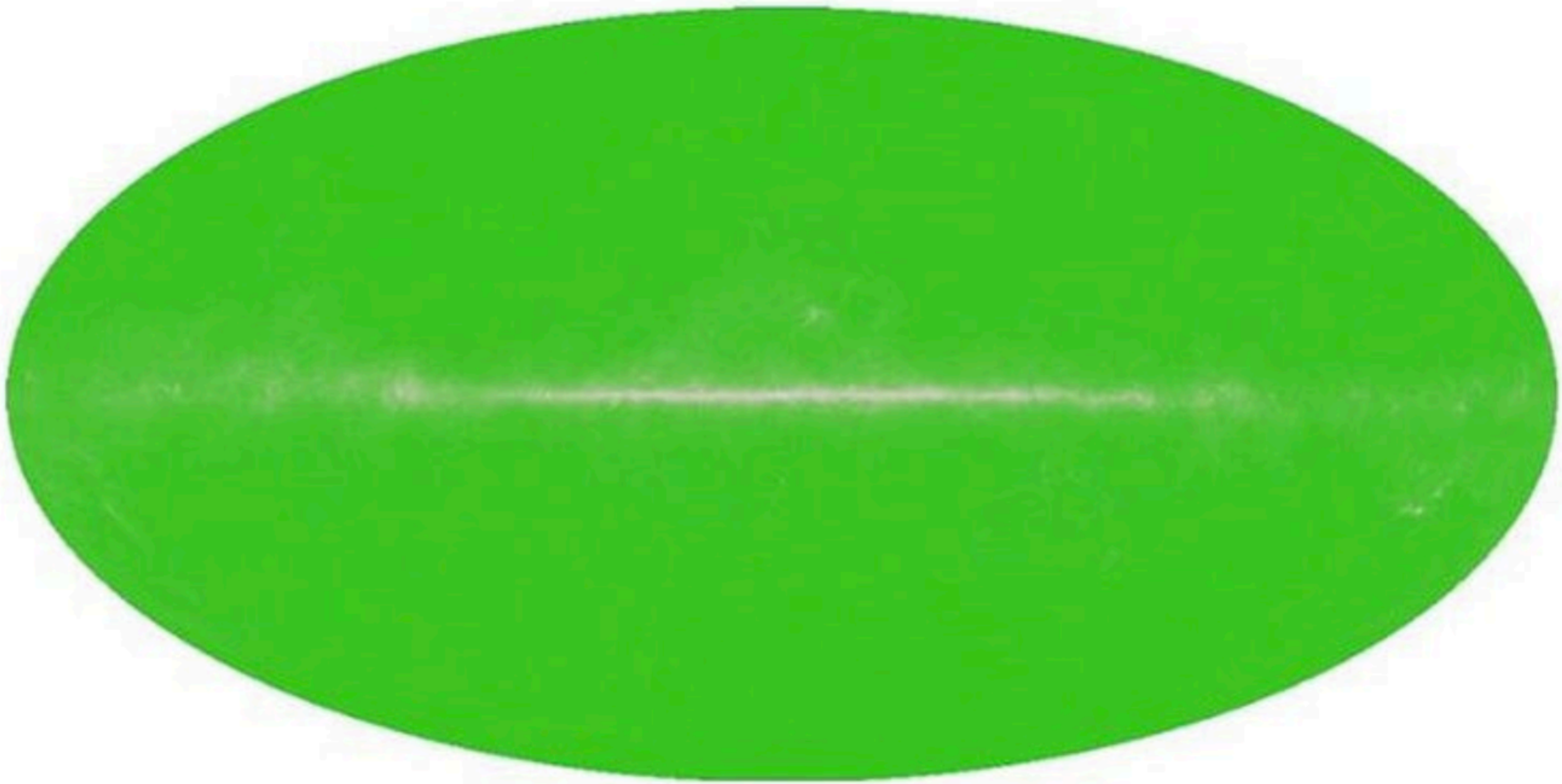
The signal observed by Penizas and Wilson was a feature predicted by hot big band theory

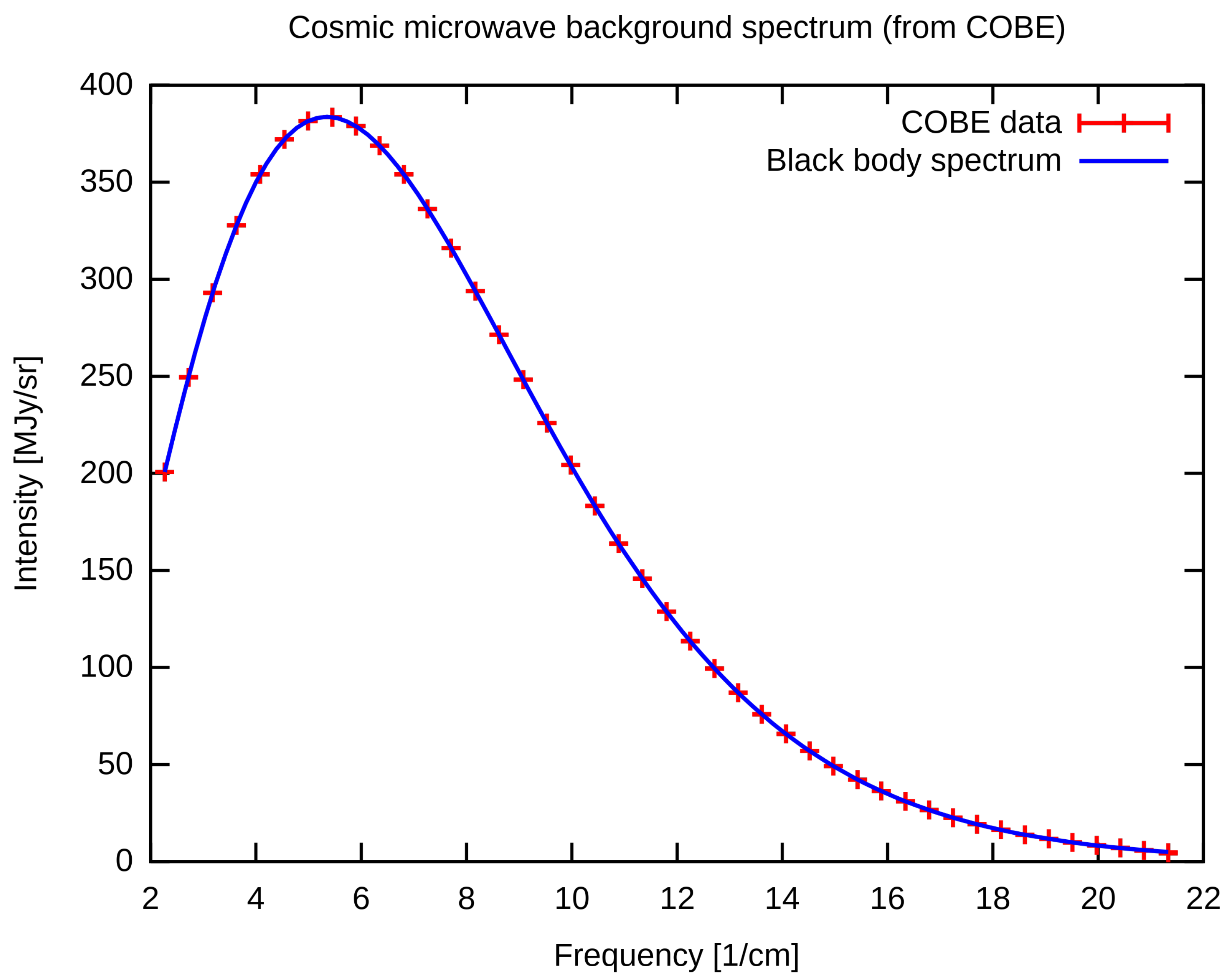


According to the original observations of Penzias and Wilson, the galactic plane emitted some astrophysical sources of radiation (center), but above and below, all that remained was a near-perfect, uniform background of radiation. (NASA / WMAP SCIENCE TEAM)

The cosmic microwave background

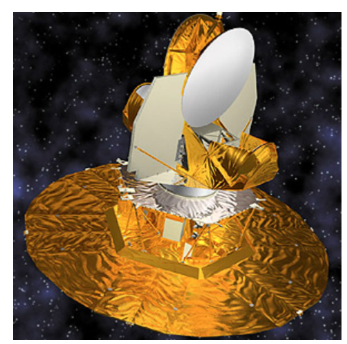
1990 – FIRAS (Far InfraRed Absolute Spectrophotometer) on the Cosmic Background Explorer (COBE) satellite measures the black body form of the CMB spectrum with exquisite precision

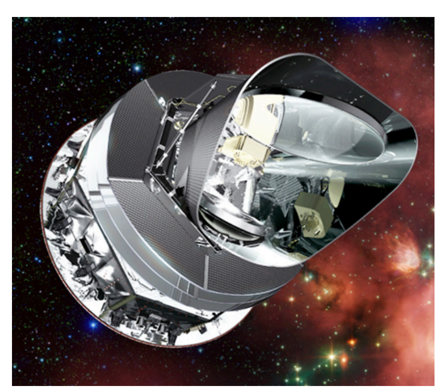
https://articles.adsabs.harvard.edu/cgi-bin/nph-iarticle_query?1990ApJ...354L..37M&classic=YES

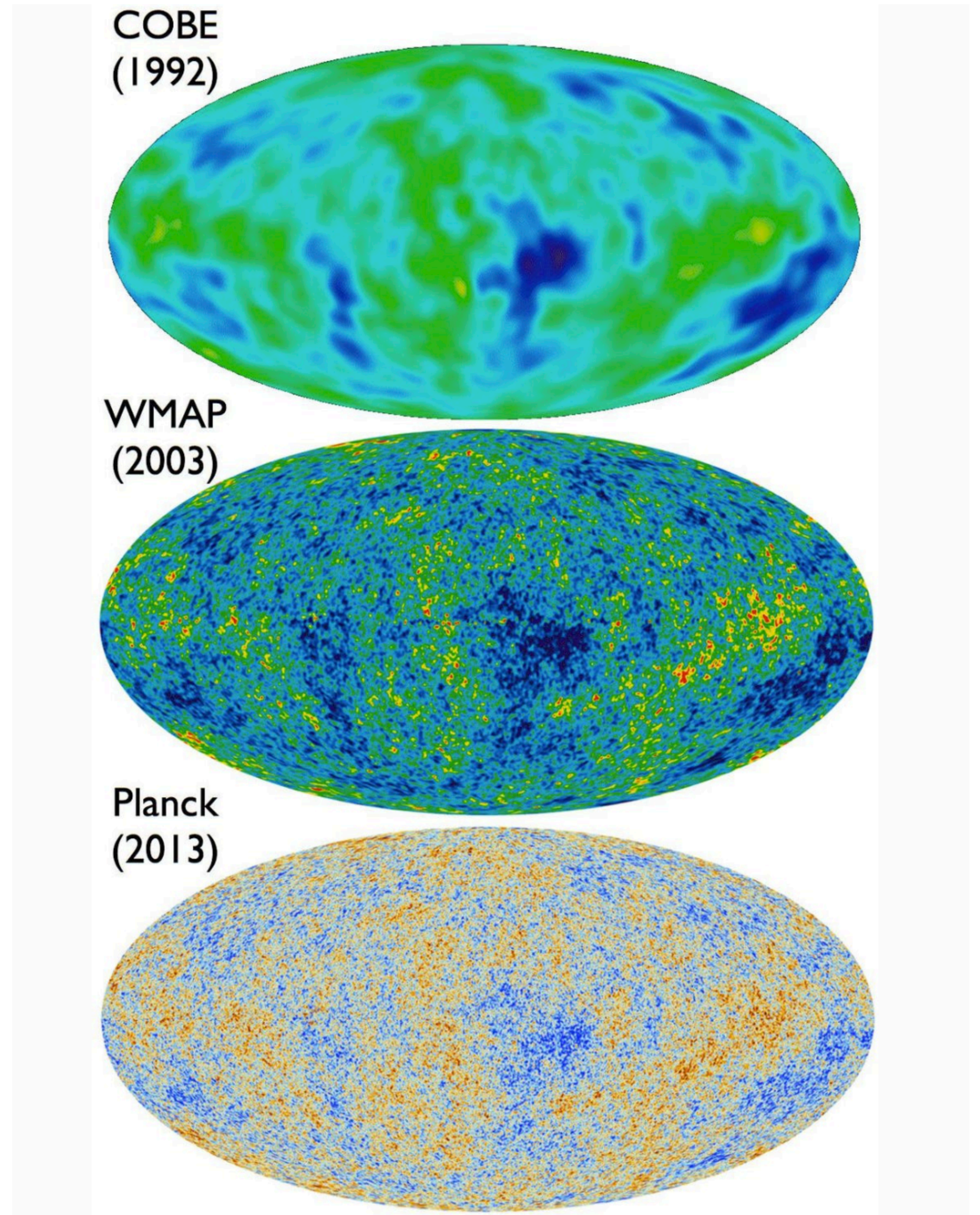


The cosmic microwave background

Since then the focus has been on measuring the anisotropies of the cosmic microwave background

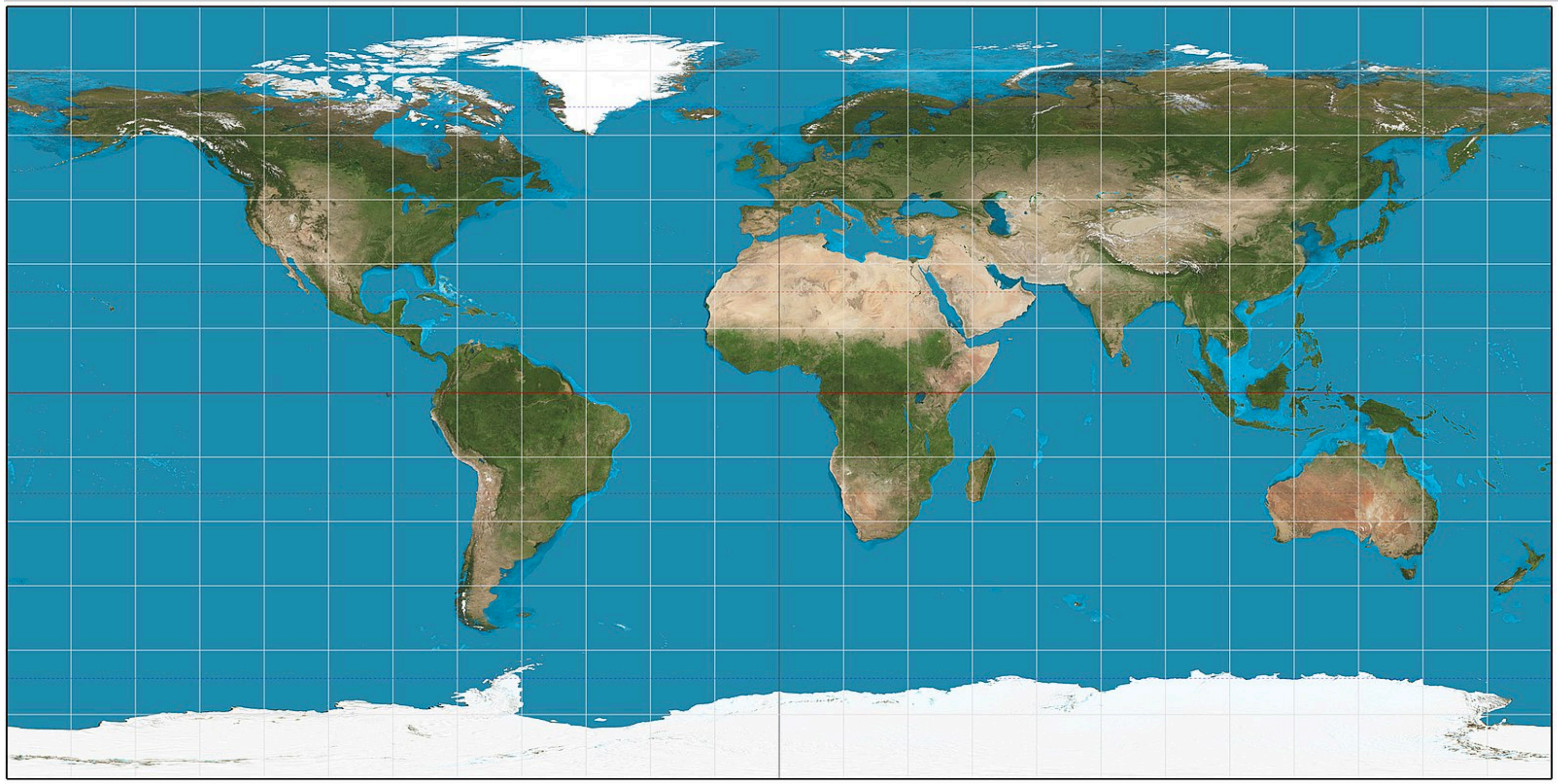


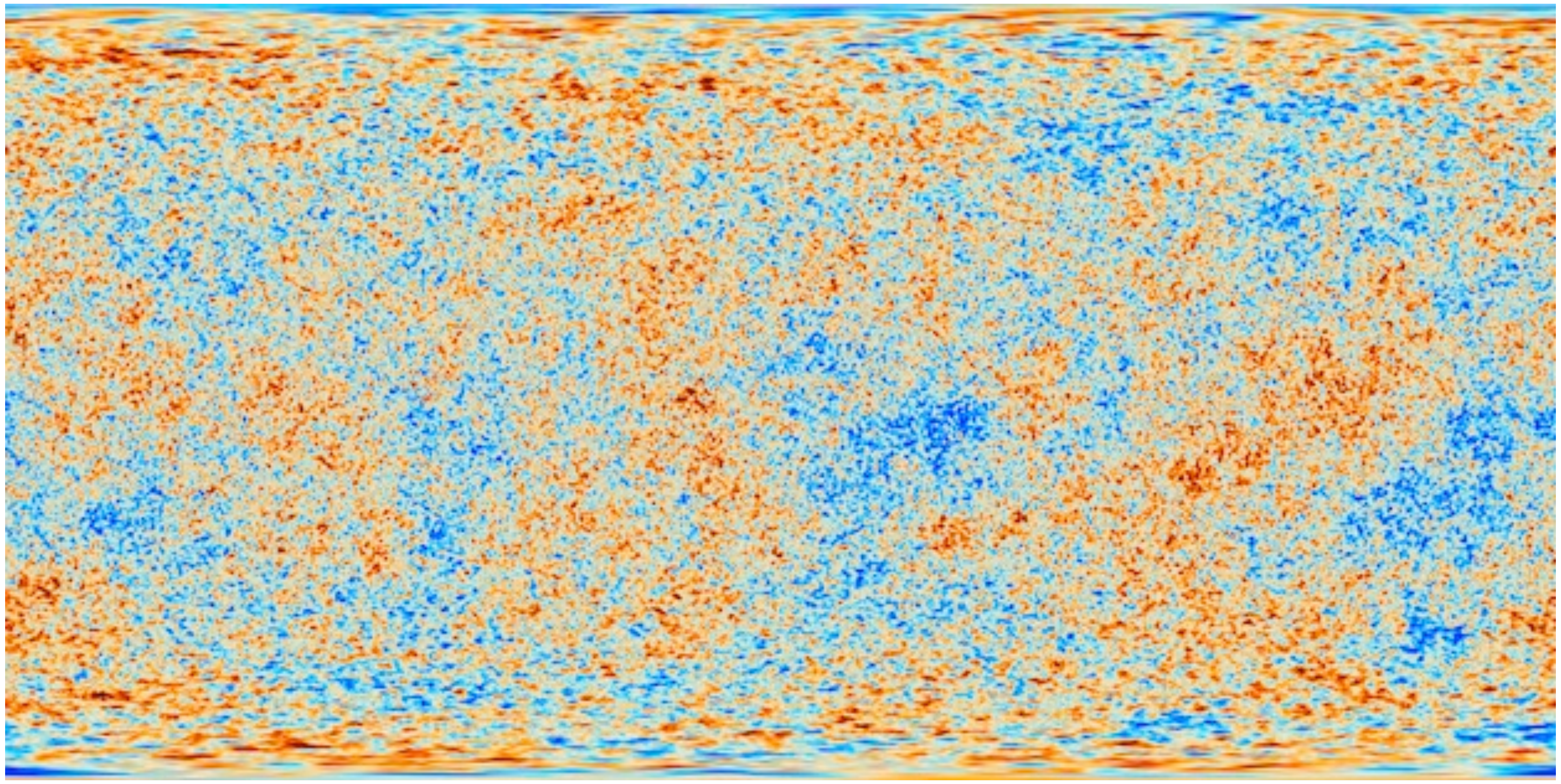




About projection

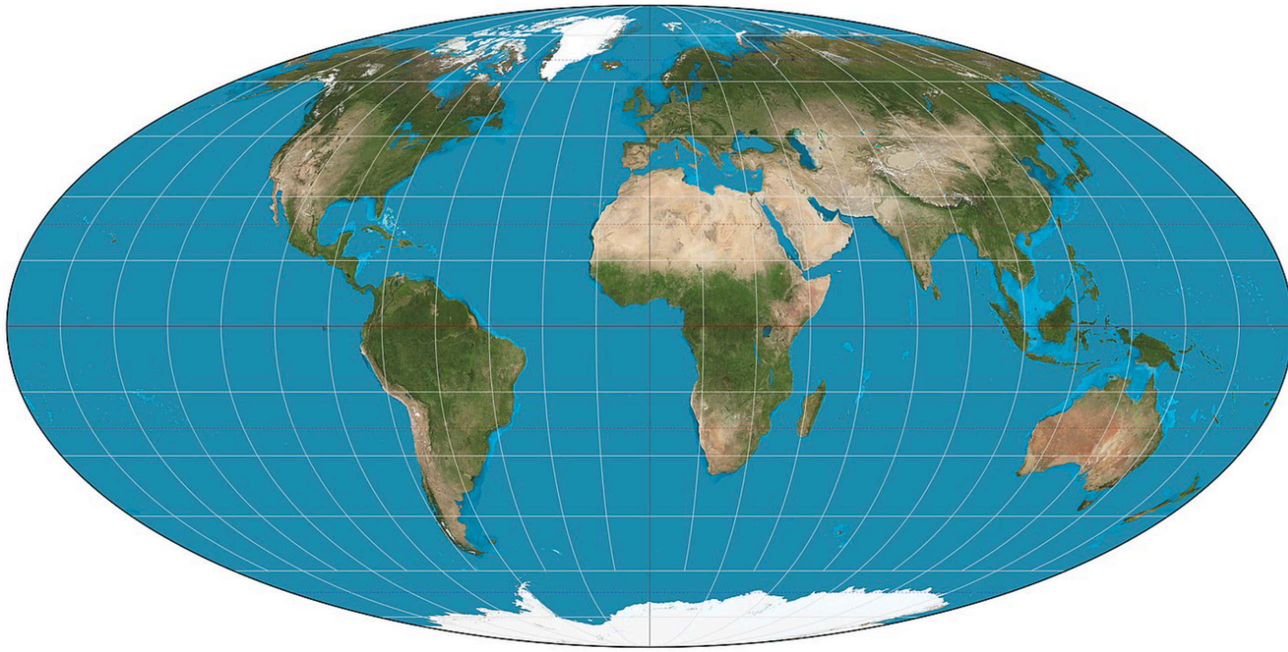
Equirectangular projection leads to strong distorsions at the poles

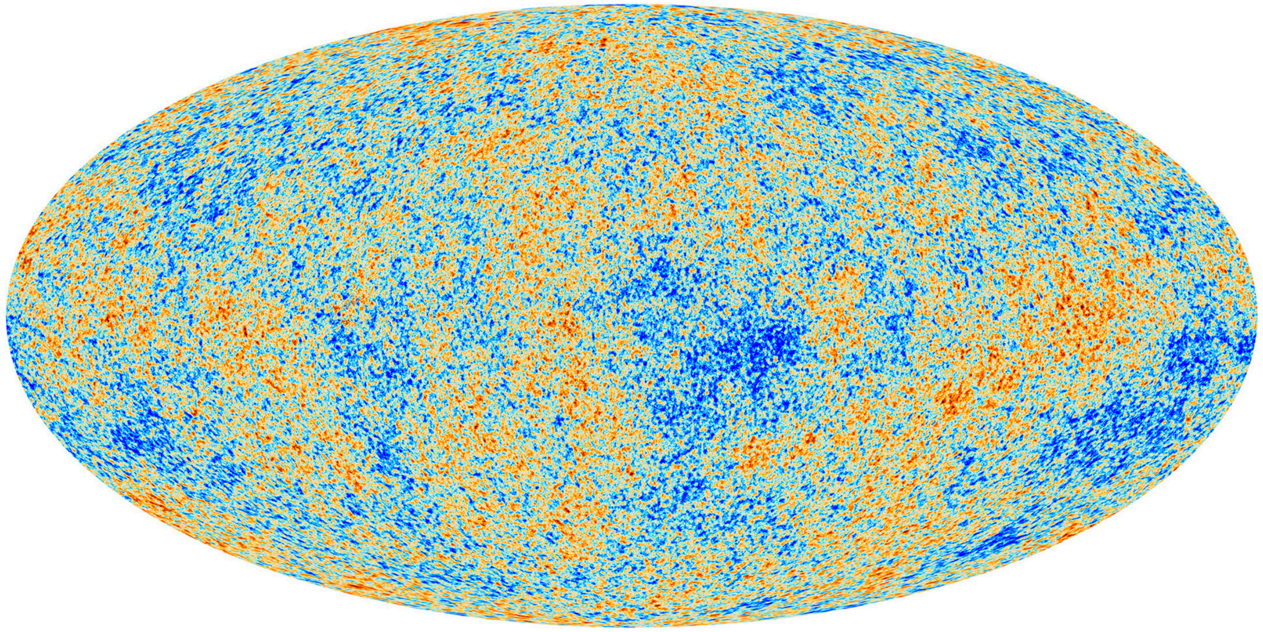


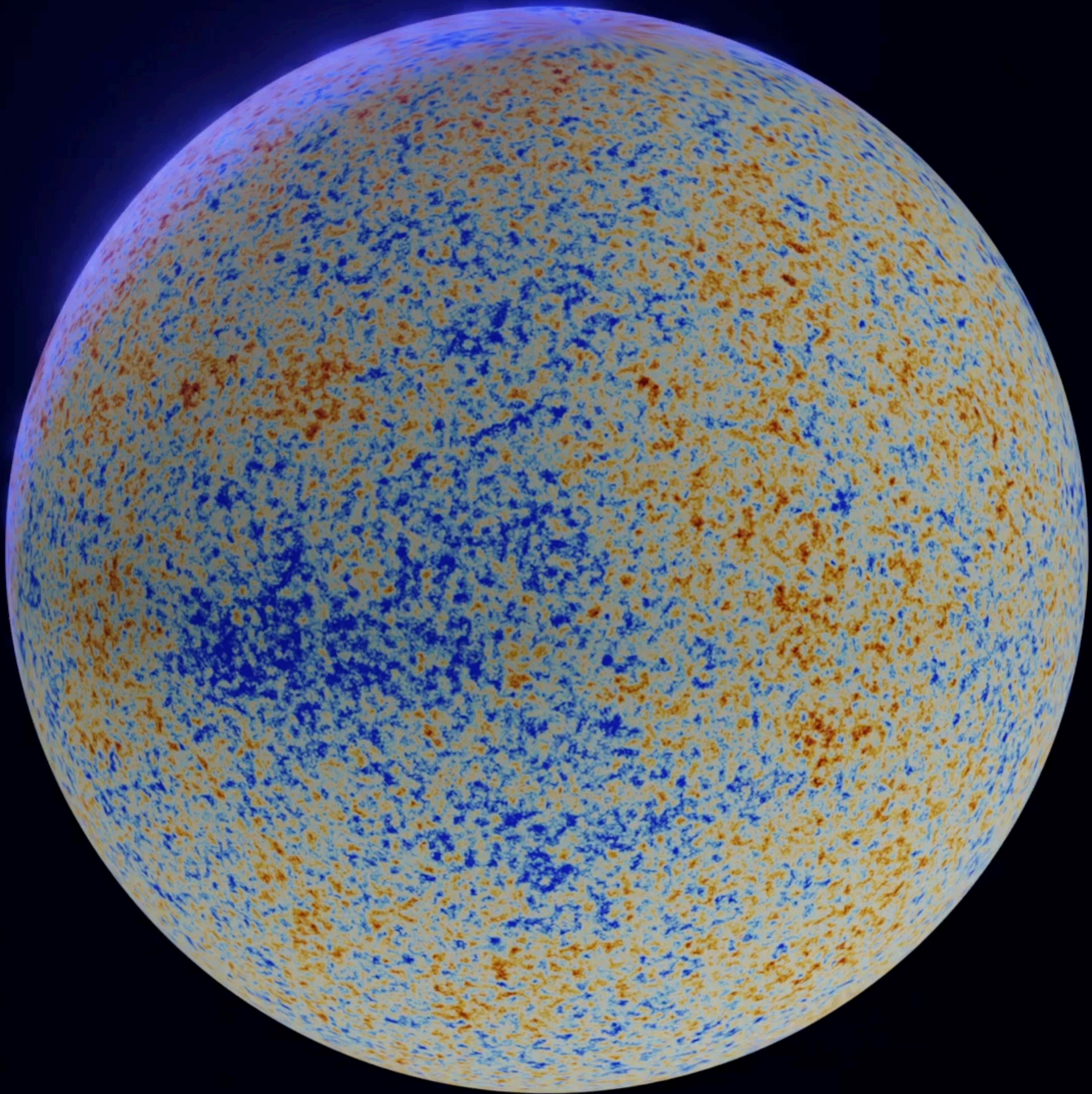


About projection

The standard in the CMB community
is to use the Mollweide projection which
preserves accuracy of proportions in area







So what do we actually measure when we look at the cosmic microwave background anisotropies?

The early universe is composed of a plasma with four different components: Photons, Neutrinos, Dark Matter, Baryons

This 4 species influence together in the following way

Photons and electrons interact via Compton scattering

$$e^- + \gamma \longleftrightarrow e^- + \gamma$$

Electrons and protons interact via Coulomb scattering

$$e^- + p \longleftrightarrow e^- + p$$

Neutrinos interact via weak interaction

$$\nu_e + e^- \quad \longleftrightarrow \quad \nu_e + e^-$$

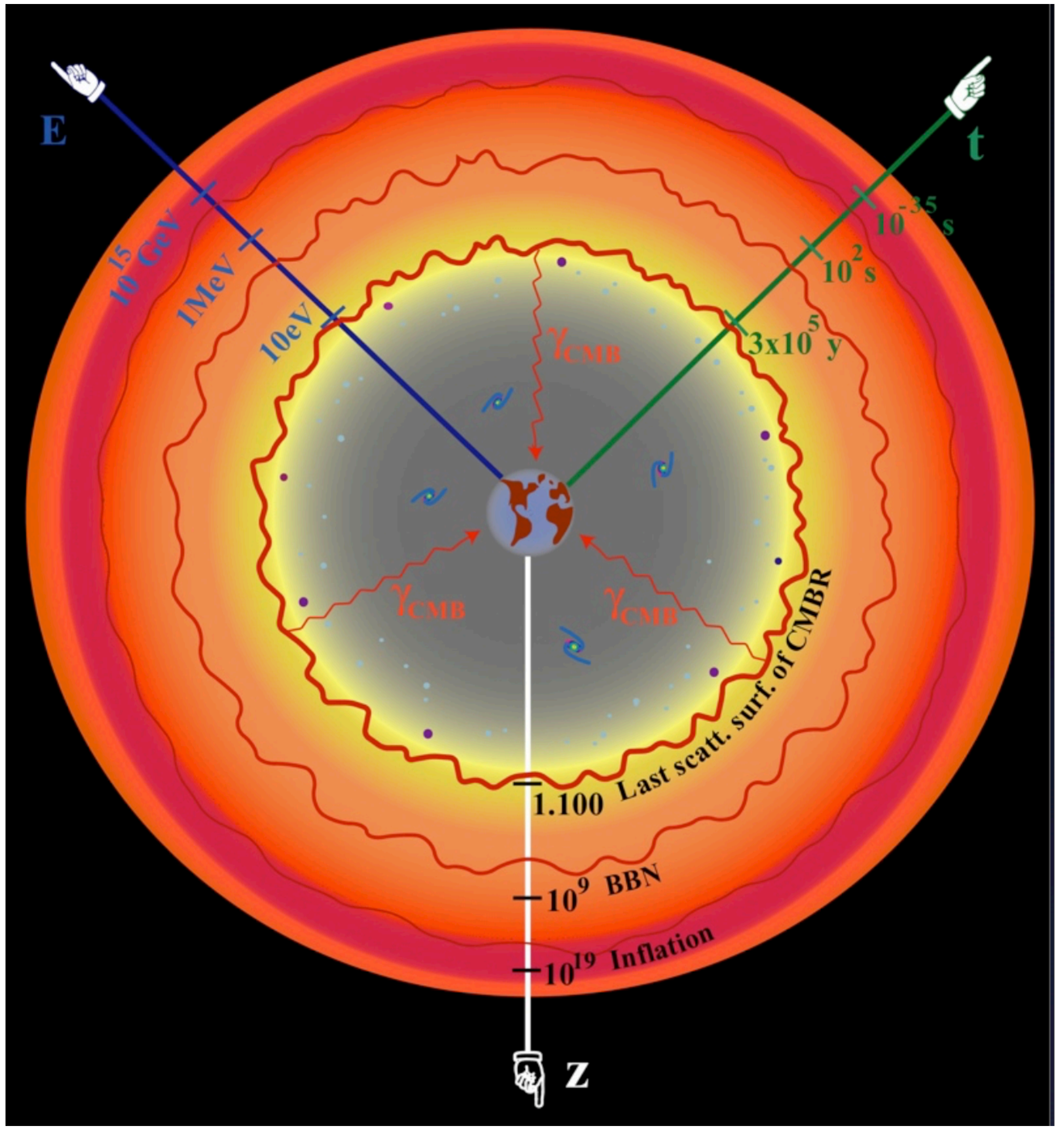
Dark matter only interacts gravitationally

At a redshift around z = 1100, T = 3 eV = 3000 K, Compton scattering of electrons and photons become inefficient and electrons and protons can recombine into hydrogen atoms (and some helium) the CMB is emitted

$$p + e^{-} \longrightarrow H + \gamma$$

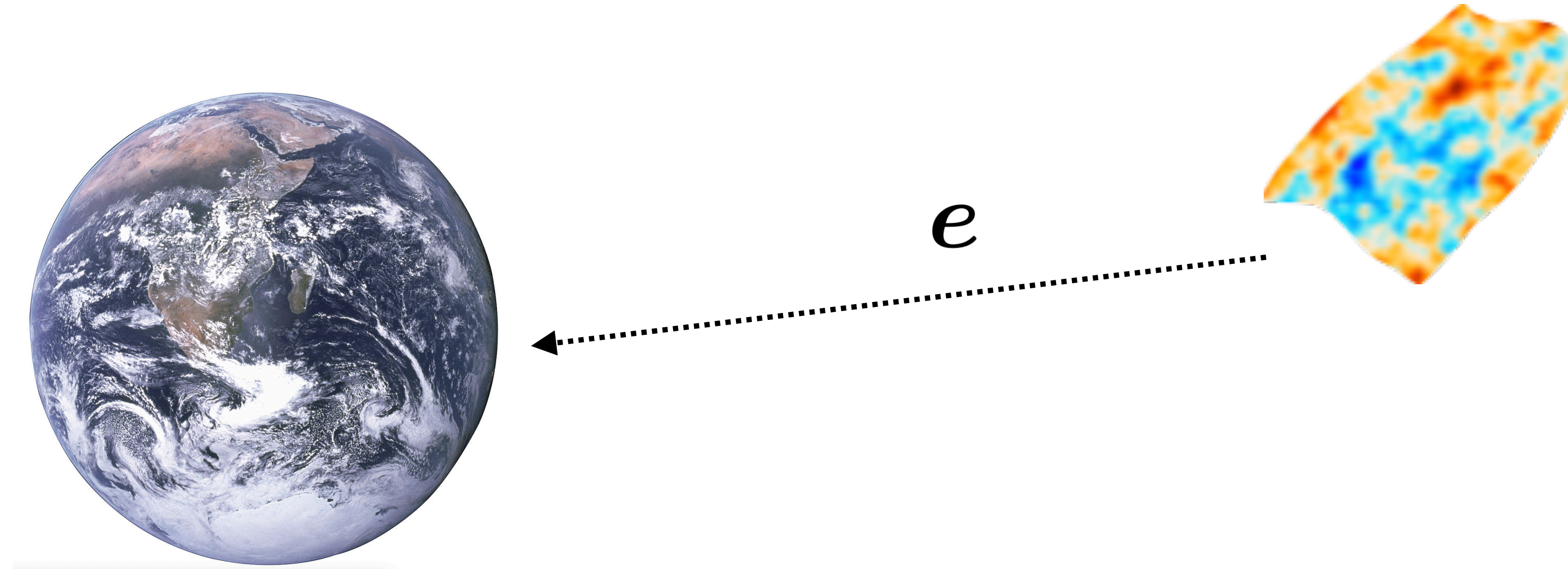
The photons free stream towards us
And we observe the average
CMB temperature today with

$$T_{z=0,\mathrm{CMB}} = \frac{T_{z=z_{\mathrm{rec}},\mathrm{CMB}}}{1+z_{\mathrm{rec}}} = 2.7\ \mathrm{K}$$



Observed CMB temperature

$\Theta = \Theta(\boldsymbol{x}_{\mathrm{Earth}}, \hat{n}, t_0)$



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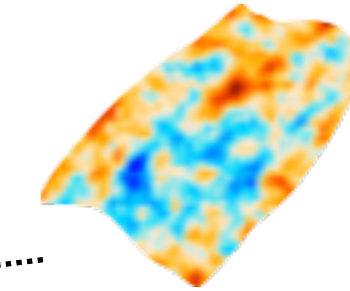

$$\Theta\Big|_{\eta_R} = (\Theta_0 + \psi)\Big|_{\eta_*} + \boldsymbol{e}\boldsymbol{u}_b|_{\eta_*} + 2\int_{\eta_*}^{\eta_R} \dot{\psi}\, d\eta$$

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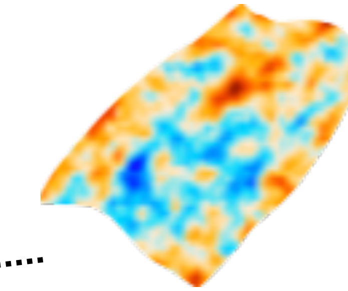

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« Monopole »: local temperature of
The CMB on the last scattering surface

$$\Theta_0(\boldsymbol{x}_{\mathrm{LSS}}, t) = \frac{1}{4\pi}\int d\hat{p}\,\Theta(\hat{p}, \boldsymbol{x}_{\mathrm{LSS}}, t)$$

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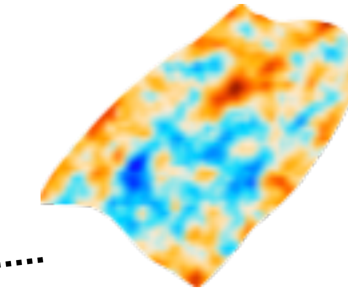

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Gravitational potential
-> Gravitational doppler effect

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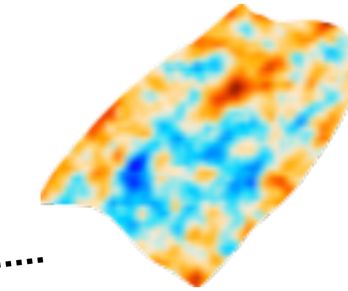

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$\boldsymbol{e}$ direction of
propagation of photons

$\boldsymbol{v}_b$ velocity of the
baryons-photons fluid at decoupling
-> Standard doppler effect

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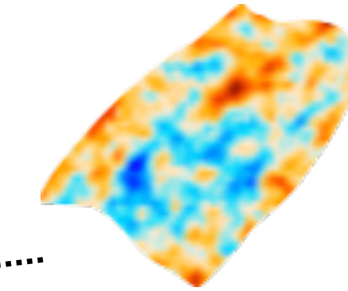

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Variations of the gravitational potential along the path of the photons from the last scattering surface to us
-> Integrated Sachs Wolfe effect

« Monopole »: local temperature of
The CMB on the last scattering surface

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The CMB maps will be the sum of the effect of :

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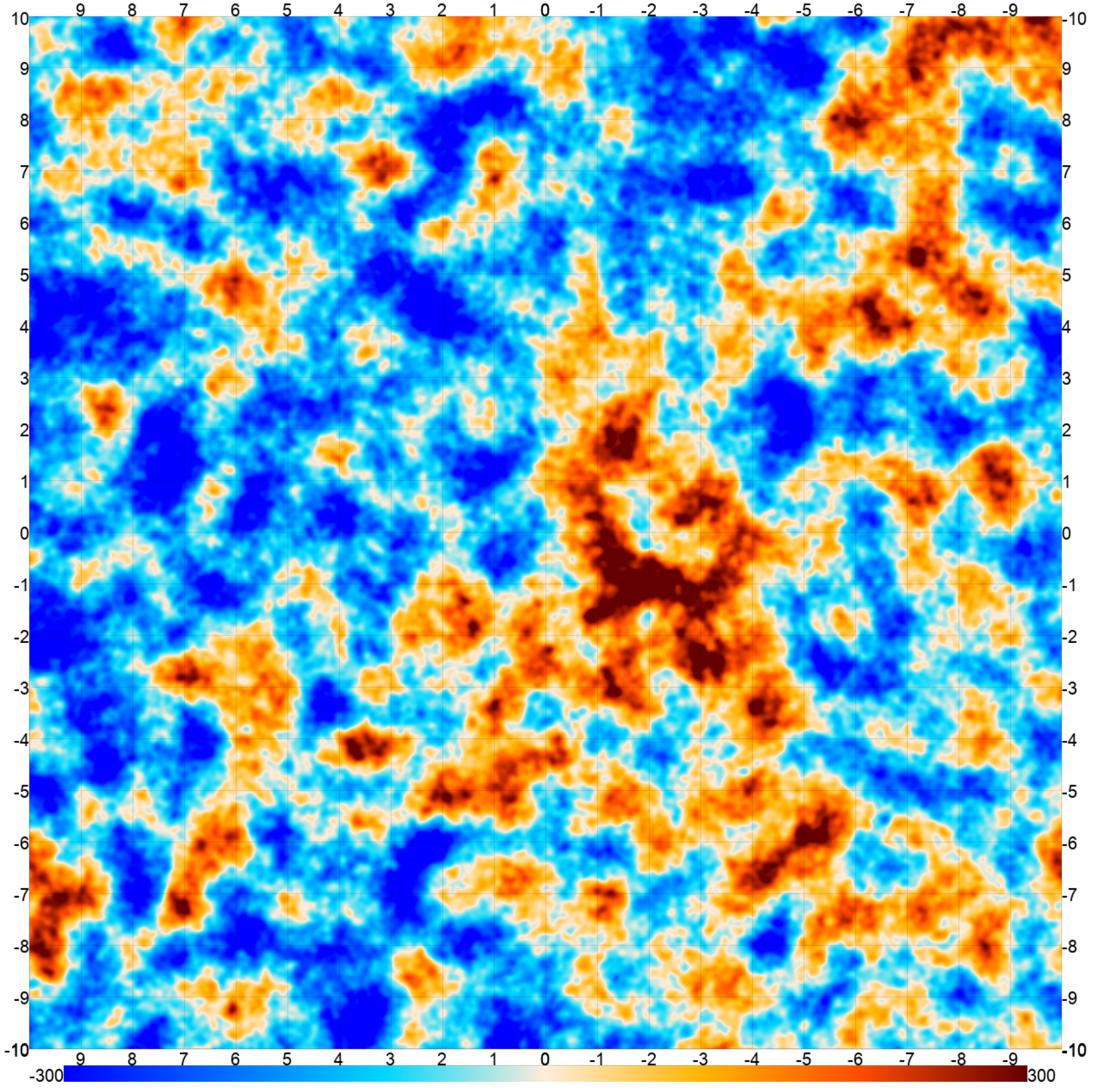
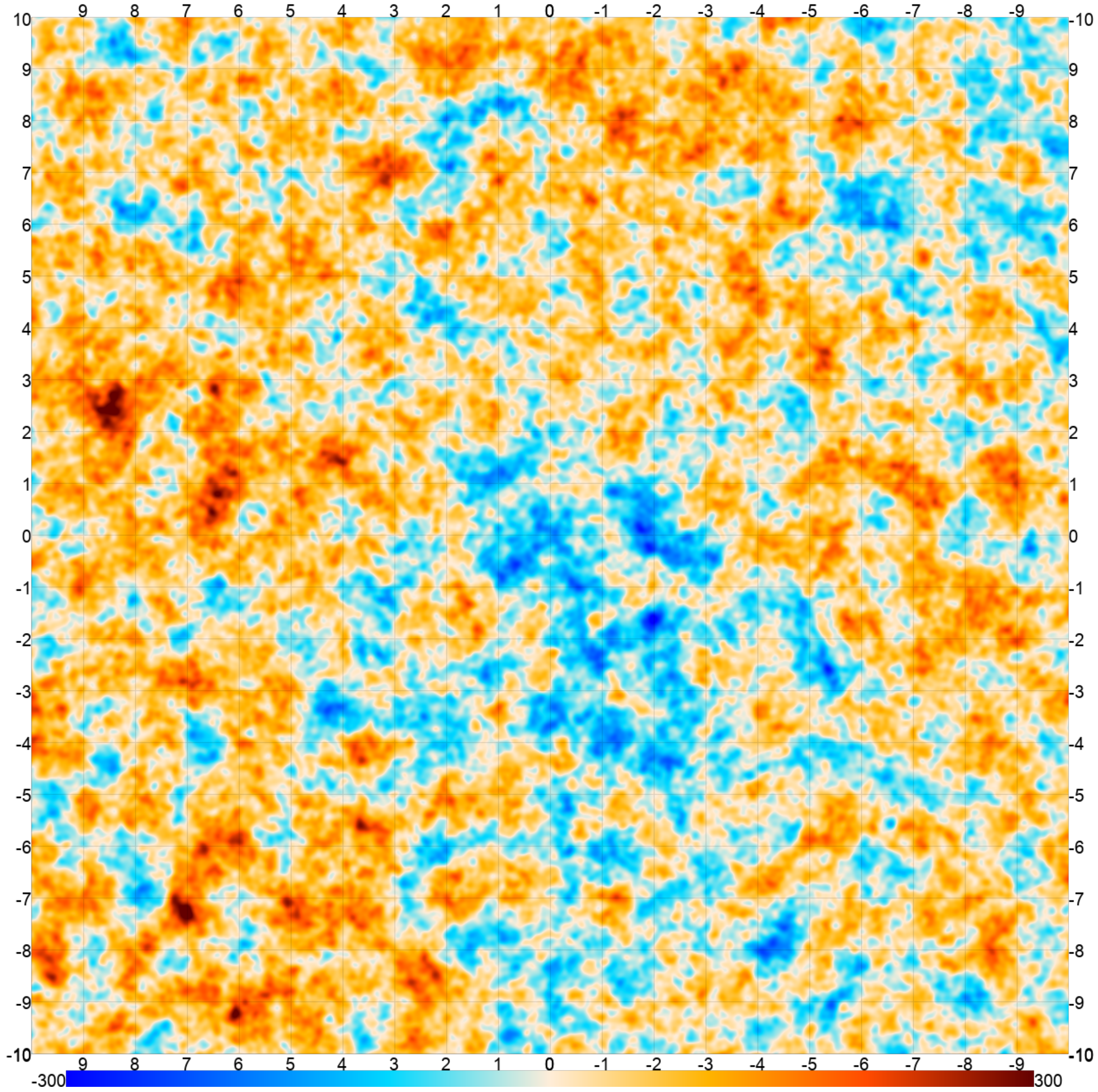


The CMB maps will be the sum of the effect of :

the local temperature on the LSS + The gravitational doppler effect



The CMB maps will be the sum of the effect of :

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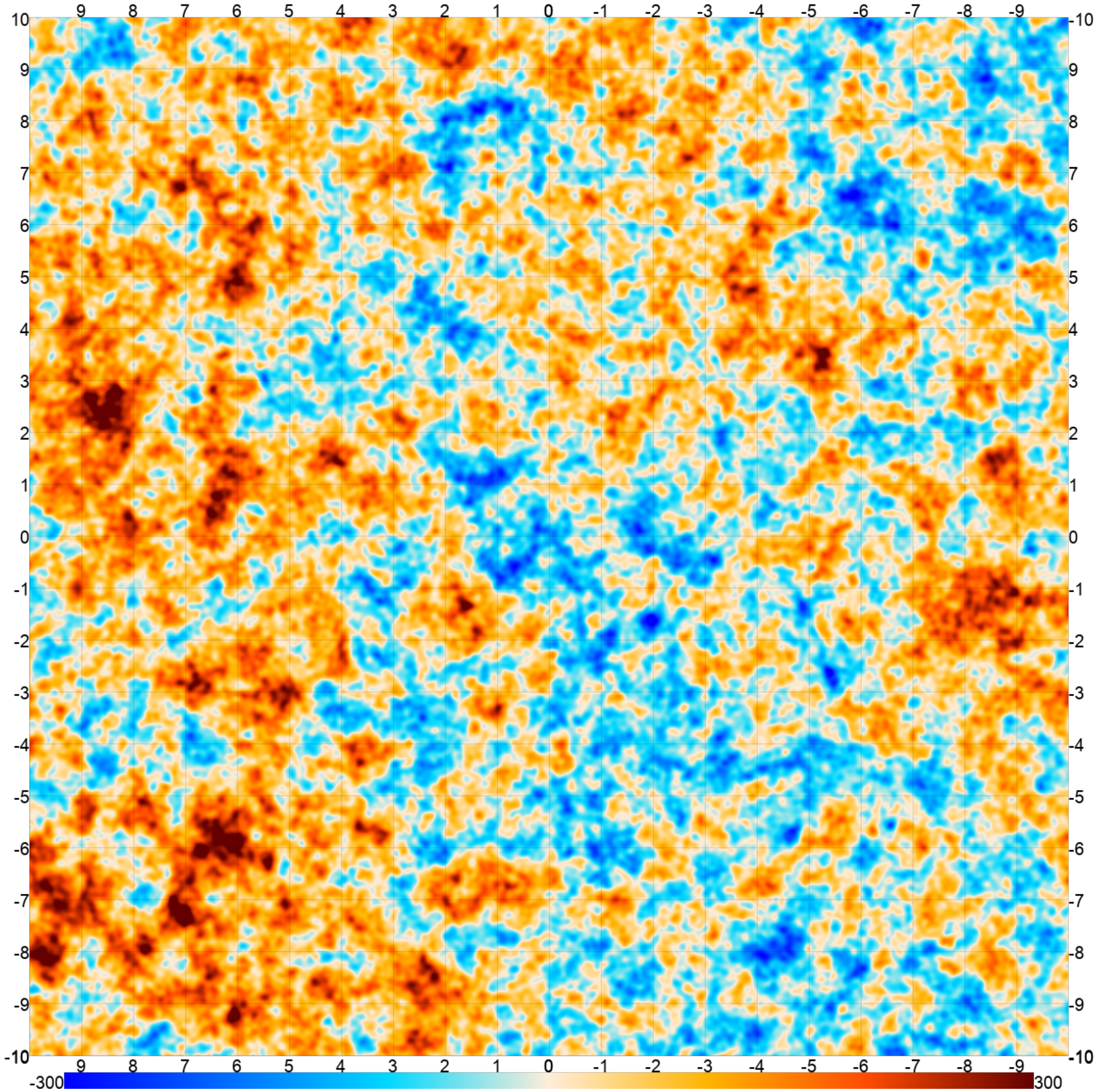
+ the kinematic doppler effect



The CMB maps will be the sum of the effect of :

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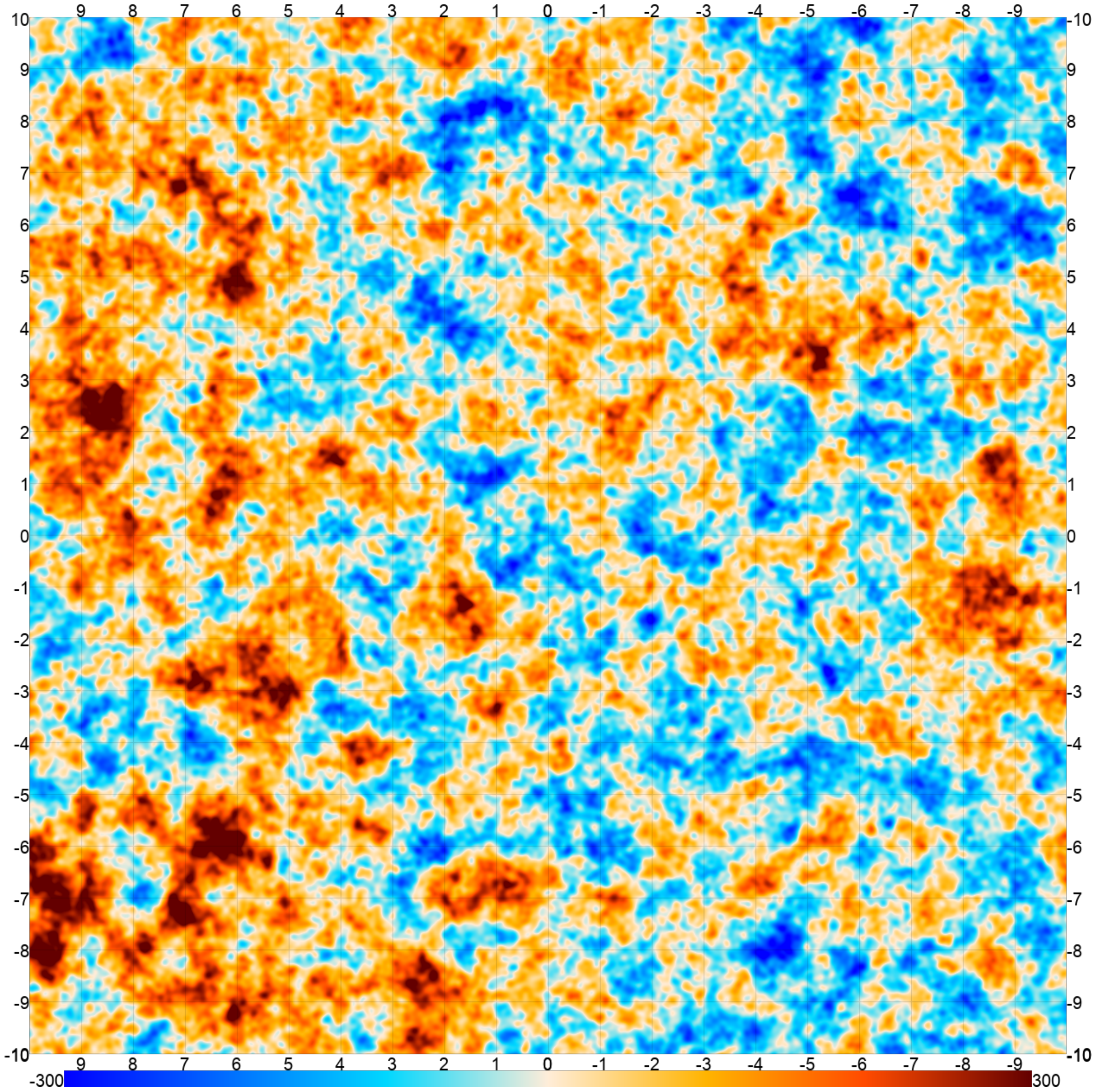
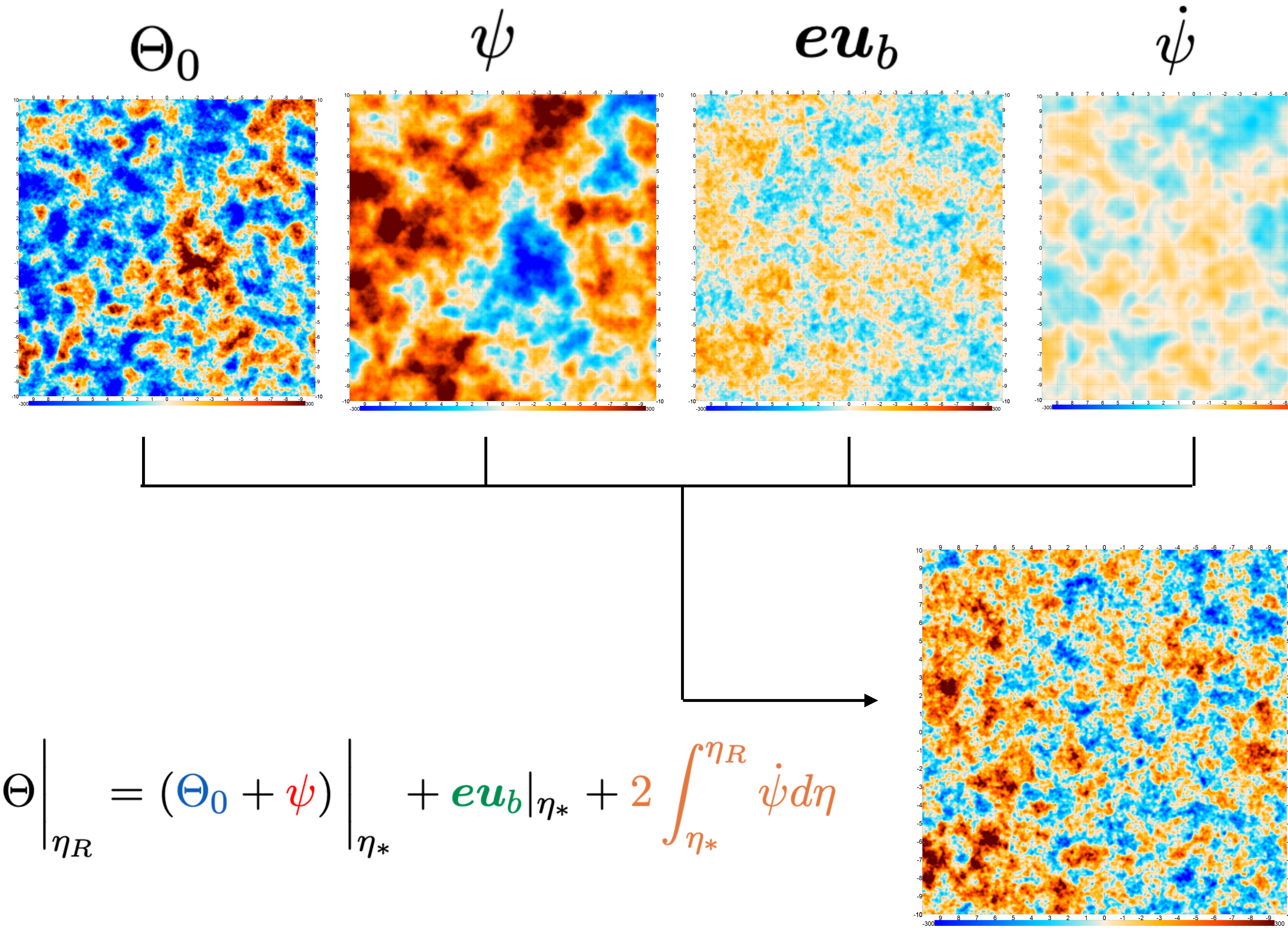
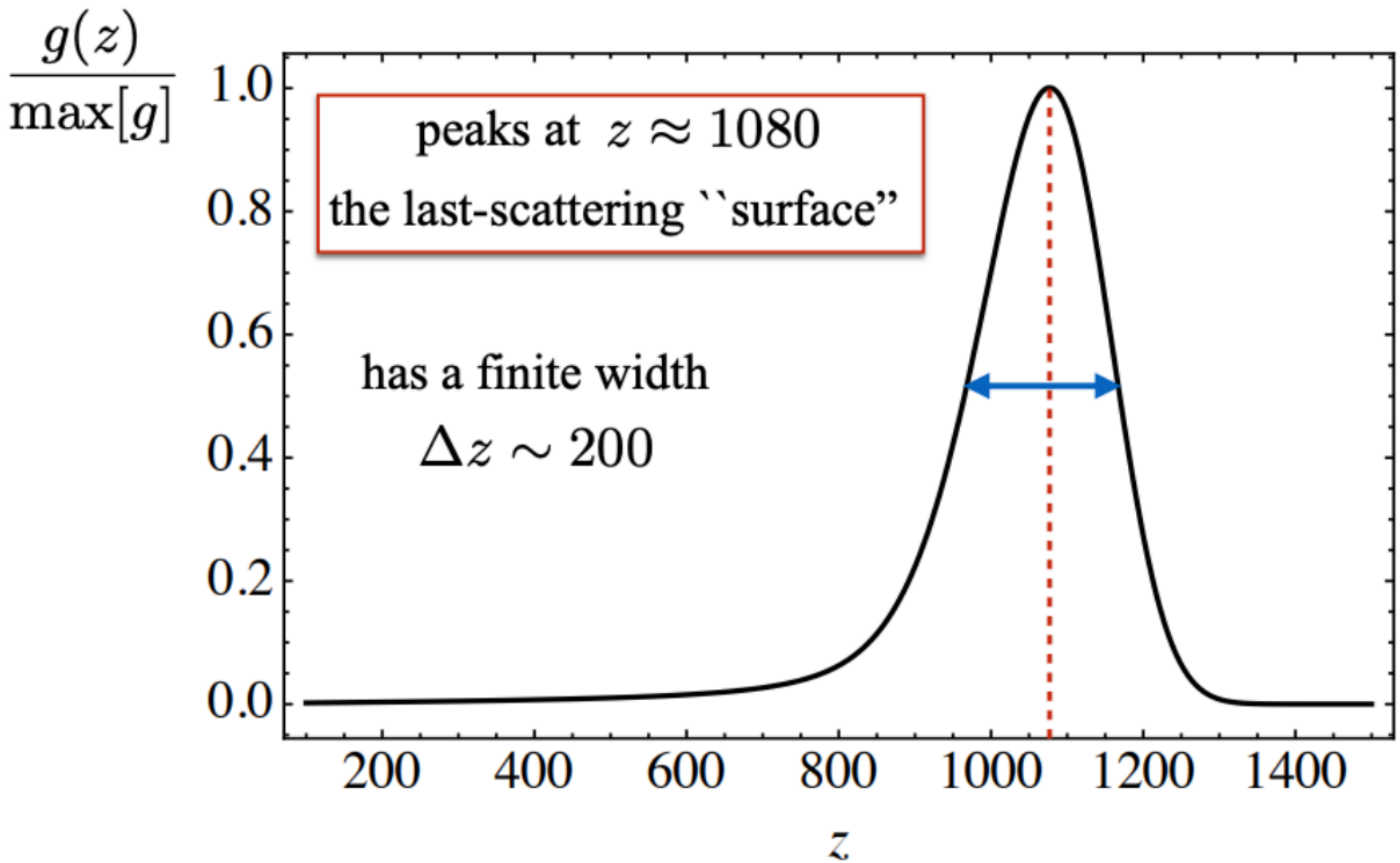
+ the kinematic doppler effect + the ISW effect





$$\Theta\Big|_{\eta_R} = (\Theta_0 + \psi)\Big|_{\eta_*} + \boldsymbol{e u}_b|_{\eta_*} + 2\int_{\eta_*}^{\eta_R} \dot{\psi}\, d\eta$$

In reality, the Universe transition from opaque to transparent is not instantaneous, So we have to introduce the visibility function.
It's a function that represent the probability that a photon last scatters at a given redshift:



The last scattering surface has a finite **thickness**

This modify our formula into

$$\boxed{\Theta\Big|_{\eta_R} = \int_0^{\eta_R} d\eta g(\eta)\left[(\Theta_0 + \psi) + \boldsymbol{e}\boldsymbol{u}_b\right] + 2\int_0^{\eta_R} d\eta \exp(-\tau)\dot{\psi}}$$

Fundamental equation of CMB anisotropies

So the right hand side is the source of the temperature anisotropies observed today

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Fundamental equation of CMB anisotropies

So the right hand side is the source of the temperature anisotropies observed today

The question become, how do we compute each terms of this expression ?
In order to compute the value of the CMB temperature today we therefore need to know:

- What is the local temperature of the plasma at the time the CMB is emitted
- What is the evolution of the gravitational potential wells in the Universe
- What is the plasma velocity
- What is the visibility function, i.e, how does the free electron density varies

The first thing we need is a metric, that tells us how stuff moves in the universe
In this course we will use the perturbed FLRW metric in Newtonian gauge

$$ds^2 = -(1 + 2\psi(\boldsymbol{x}, t))dt^2 + a^2(t)\delta_{ij}[1 + 2\phi(\boldsymbol{x}, t)]dx^i dx^j$$

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At the time of CMB emission $\phi, \psi \sim 10^{-4}$

so any higher order term can be dropped

Then we use <u>two set of equations</u>:

The first set tells us how gravity responds to the matter-energy content of the Universe, there are the Einstein equations, they tell us about the evolution of the gravitational potential wells in the Universe

$$k^2\phi + 3\frac{a'}{a}\left(\phi' - \psi\frac{a'}{a}\right) = 4\pi G a^2 \delta\rho$$

$$k^2(\phi + \psi) \sim 0$$

Einstein equations

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$$k^2(\phi + \psi) \sim 0$$

This equation is the generalization of the Poisson equation in general relativity in the newtonian limit, it take the familiar form

$$-\nabla^2\phi = 4\pi G a^2 \delta\rho \qquad \delta\rho = \rho_c\delta_c + \rho_b\delta_b + \rho_\gamma\delta_\gamma + \rho_\nu\delta_\nu$$

The extra terms are there to take into account the expansion of the universe

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$$\boxed{k^2(\phi + \psi) \sim 0}$$

This equation tell us that the two potentials are approximately equal in magnitude and opposite in sign

The second set tells us about how the perturbations evolves, and are called the Boltzmann equations

These equations tell us about the dynamics of each components of the plasma

$$\begin{aligned}
\Theta' + ik\mu\,\Theta &= -\phi' - ik\mu\psi - \tau'\left[\Theta_0 - \Theta + \mu u_b - \frac{1}{2}P_2\mu\Pi\right] \\
\delta_c' + iku_c &= -3\phi' \\
u_c' + \frac{a'}{a}u_c &= -ik\,\psi \\
\delta_b' + iku_b &= -3\phi' \\
u_b' + \frac{a'}{a}u_b &= -ik\,\psi + \frac{\tau'}{R}[u_b + 3i\Theta_1]
\end{aligned}$$

δ_c, δ_b Dark matter and baryons overdensity

u_c, u_b Dark matter and baryons velocity

While it would take too long to re-derive all of these equations here, we can gain some intuition by re-deriving the ones followed by dark matter.

The main tool for deriving Boltzmann equation is the distribution function.
Let's consider a set of particules occupying some region of space,
These particles are completely described by their positions and momenta $\{\boldsymbol{x_i}, \boldsymbol{p_i}\}$

We can define a distribution function $f(\boldsymbol{x}, \boldsymbol{p}, t)$ which relate to the number of particles in a small phase space elements around $(\boldsymbol{x}, \boldsymbol{p})$

$$N(\boldsymbol{x}, \boldsymbol{p}, t) = f(\boldsymbol{x}, \boldsymbol{p}, t)(\Delta x)^3 \frac{(\Delta p)^3}{(2\pi)^3}$$

Number of particles at position x with momentum p at time t

Distribution function

Volume of phase space element

The distribution function can be used to define all macroscopic properties of a collection of particules, e.g the density and energy density

$$n_s(\boldsymbol{x},t) = g_s \int \frac{d^3p}{(2\pi)^3} f_s(\boldsymbol{x},\boldsymbol{p},t)$$

$$\rho_s(\boldsymbol{x},t) = g_s \int \frac{d^3p}{(2\pi)^3} f_s(\boldsymbol{x},\boldsymbol{p},t) E_s(p)$$

$$E_s(p) = \sqrt{p^2 + m_s^2}$$

We can play a bit with these equations, for example by calculating the average number and energy density of CMB photons

For example : Photons are bosons, in equilibrium at temperature T,
They have the following Bose Einstein distribution function

$$f_\gamma(p;T_{\rm CMB}) = \frac{1}{\exp\left(\frac{p}{T_{\rm CMB}}\right) - 1}$$

The average density of CMB photons in the universe is given by

$$\begin{aligned} n_\gamma(T_{\rm CMB}) &= 2\int \frac{d^3p}{(2\pi)^3}\frac{1}{e^{p/T_{\rm CMB}} - 1} \\ &= \frac{8\pi T_{\rm CMB}^3}{(2\pi)^3}\int_0^\infty dx \frac{x^2}{e^x - 1} = \frac{\Gamma(3)\zeta(3)T_{\rm CMB}^3}{\pi^2} = \frac{2\zeta(3)T_{\rm CMB}^3}{\pi^2} \end{aligned}$$

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Riemman zeta function $\zeta(3) \sim 1.202$

For T=2.7 K, this give us 400 photons/cube centimeter

The Boltzmann equation simply tell us that in the absence of interaction the number of particle is conserved and we have

$$\frac{df(\boldsymbol{x},\boldsymbol{p},t)}{dt}=0$$

In case where the is interaction the equation is modified by the introduction of a collision term

$$\frac{df(\boldsymbol{x},\boldsymbol{p},t)}{dt}=C[f]$$

Let's derive the Boltzmann equation follow by dark matter, starting with the evolution of the homogeneous dark matter density, that's the simplest equation since the dark matter is supposedly non interacting,
the general relativistic version of the equation is given by

$$\frac{df(x^\mu,P^\mu)}{dt}=0$$

Where P^μ is the 4-momentum which is given in the FLRW universe

$$ds^2=-dt^2+a^2(dx^2+dy^2+dz^2)$$

$$P^\mu=\begin{pmatrix}P^0\\P^i\end{pmatrix}=\begin{pmatrix}\sqrt{p^2+m^2}\\\frac{p}{a}\hat{p}^i\end{pmatrix}$$

Let’s start by expanding the total derivative into a set of partial derivative

$$\frac{df(x^\mu,p^\mu)}{dt} = \frac{\partial f(x^\mu,p^\mu)}{\partial t} + \frac{\partial f(x^\mu,p^\mu)}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f(x^\mu,p^\mu)}{\partial p}\frac{dp}{dt} + \frac{\partial f(x^\mu,p^\mu)}{\partial \hat{p}_i}\frac{d\hat{p}_i}{dt}$$

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The equation simplify to $$\frac{df(x^\mu,p^\mu)}{dt} = \frac{\partial f(x^\mu,p^\mu)}{\partial t} + \frac{\partial f(x^\mu,p^\mu)}{\partial p}\frac{dp}{dt}$$

The only thing left to calculate is $\frac{dp}{dt}$ which account for how the momentum of a particle change with time in an expanding Universe

The evolution of the momentum with time can be computed from the time component of the geodesic equation

$$\frac{dP^\mu}{d\lambda} = -\Gamma^\mu_{\alpha,\beta} P^\alpha P^\beta$$

Leading to $\dfrac{dp}{dt} = -Hp$

Where $H = \dfrac{\dot{a}}{a}$ is the expansion rate of the universe

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The meaning of this equation is to tell us that the physical momenta of particule Decays as 1/a in an expanding universe, indeed:

$$\begin{aligned} \frac{\dot{p}}{p} &= -\frac{\dot{a}}{a} \\ d\ln(p)/dt &= -d\ln(a)/dt \\ \ln(p) &= -\ln(a) + C = \ln(a^{-1}) + C \\ p &\propto 1/a \end{aligned}$$

The Boltzmann equation followed by homogeneous dark matter is therefore

$$\begin{aligned} \frac{\partial f(x^\mu, p^\mu)}{\partial t} + \frac{\partial f(x^\mu, p^\mu)}{\partial p}\frac{dp}{dt} &= 0 \\ \frac{\partial f(x^\mu, p^\mu)}{\partial t} - Hp\frac{\partial f(x^\mu, p^\mu)}{\partial p} &= 0 \end{aligned}$$

We can integrate this over momentum in order to get an evolution equation for The density of dark matter in the Universe

$$\begin{aligned} \frac{\partial \int \frac{d^3p}{(2\pi)^3} f}{\partial t} - H\int \frac{d^3p}{(2\pi)^3} p\frac{\partial f}{\partial p} &= 0 \\ \frac{\partial n}{\partial t} - H\int \frac{d^2\hat{p}}{(2\pi)^3}\int dp p^3\frac{\partial f}{\partial p} &= 0 \\ \frac{dn}{dt} + 3Hn &= 0 \end{aligned}$$

Where the last equality follow from integration by part

This is a familiar result, the number density of matter decrease as the cube Of the scale factor in an expanding Universe $\frac{d\ln n}{dt} = -3\frac{d\ln a}{dt} \longrightarrow n \propto a^{-3}$

Ok so this was for the background, but what about the dynamics of dark matter perturbations?, we need to go back to our original Boltzmann equation, now keeping All the terms (since perturbation unlike background can depends on $(\boldsymbol{x}, \hat{p})$

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To compute the rest of the total derivatives we need the perturbed FLRW metric,

$$ds^2 = -(1 + 2\psi(\boldsymbol{x}, t))dt^2 + a^2(t)\delta_{ij}[1 + 2\phi(\boldsymbol{x}, t)]dx^i dx^j$$

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After some work we get

$$\begin{aligned} \frac{dx^i}{dt} &= \frac{p}{aE}(1 - \phi + \psi)\hat{p}^i \\ \frac{dp}{dt} &= -[H + \dot{\phi}]p - \frac{E}{a}\hat{p}^i\frac{\partial \psi}{\partial x^i} \end{aligned}$$

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\end{aligned}$$

Compare this with the equation for the background

$$\frac{dp}{dt} = -Hp$$

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This term tell us that particule gain momentum when they fall in a gravitational potential well ψ

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Ok nearly done, the Boltzmann equation for dark matter perturbations become

$$\frac{\partial f_c}{\partial t} + \frac{\partial f_c}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_c}{\partial p}\frac{dp}{dt} = 0$$
$$\frac{\partial f_c}{\partial t} + \frac{\partial f_c}{\partial x^i}\frac{p}{aE}(1-\phi+\psi)\hat{p}^i - \left[(H+\dot{\phi})p + \frac{E}{a}\hat{p}^i\frac{\partial \psi}{\partial x^i}\right]\frac{\partial f_c}{\partial p} = 0$$

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\end{aligned}
$$

We can simplify one step further by noticing that $\frac{\partial f_c}{\partial x^i}$ is of order 1 in perturbation, since the homogeneous distribution function do not depend on positions. We finally arrive at the collisionless Boltzmann equation that describe the evolution of dark matter perturbation

$$
\frac{\partial f_c}{\partial t} + \frac{\partial f_c}{\partial x^i}\frac{p}{aE}\hat{p}^i - \left[(H+\dot{\phi})p + \frac{E}{a}\hat{p}^i\frac{\partial \psi}{\partial x^i}\right]\frac{\partial f_c}{\partial p} = 0
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Note that here we don't know anything about f_c,

since we don't know. anything about dark matter, apart from the fact that it's something non-interacting. Yet it doesn't matter, we can integrate the equation over momenta to turn it into an equation for dark matter perturbation, the result of the integration is called the « 0th order moment of Boltzmann equation »

$$\frac{\partial}{\partial t}\int\frac{d^3p}{(2\pi)^3}f_c+\frac{1}{a}\frac{\partial}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}f_c\frac{p}{E(p)}\hat{p}^i \quad - \quad (H+\dot{\phi})\int\frac{d^3p}{(2\pi)^3}p\frac{\partial f_c}{\partial p}$$
$$- \quad \boxed{\frac{1}{a}\frac{\partial\psi}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}\frac{\partial f_c}{\partial p}E(p)\hat{p}^i} = 0$$

This term is second order, because only the anisotropic of f contribute to the integral, so the integral is first order,

the multiplication by $\frac{\partial\psi}{\partial x^i}$ make it second order

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$$\frac{\partial n_c}{\partial t}+\frac{1}{a}\frac{\partial}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}f_c\frac{p}{E(p)}\hat{p}^i+3(H+\dot{\phi})n_c=0$$

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The second term under the integral is nothing but the definition of the fluid velocities

$$u_c^i=\frac{1}{n_c}\int\frac{d^3p}{(2\pi)^3}f_c\frac{p}{E(p)}\hat{p}^i$$

$$\boxed{v^i=\frac{1}{n}\left\langle\frac{p^i}{m}\right\rangle\simeq\frac{1}{n}\left\langle\frac{p\hat{p}^i}{E}\right\rangle}$$

Ok let's stop here for a minute, we have derived an equation for the evolution of the dark matter density in a FLRW universe, including the effect of perturbation

$$\frac{\partial n_c}{\partial t} + \frac{1}{a}\,\frac{\partial}{\partial x^i}(n_c u_c^i) + 3(H + \dot{\phi})n_c = 0$$

The equation should look familiar it is the generalization of the standard continuity equation in fluid dynamic

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

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We can decompose the equation into two equations, one for the background one For the perturbation using $n_c(\boldsymbol{x}, t) = \bar{n}_c(t)[1 + \delta_c(\boldsymbol{x}, t)]$

$$\frac{d\bar{n}_c}{dt} + 3H\bar{n}_c = 0$$

$$\frac{\partial \delta_c}{\partial t} + \frac{1}{a}\frac{\partial u_c^i}{\partial x^i} + 3\dot{\phi} = 0$$

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We have an equation for $\delta_c(\boldsymbol{x}, t)$ but it depends on $\boldsymbol{u}_c(\boldsymbol{x}, t)$
for solving the evolution of dark matter we therefore
need a additional equation for $\boldsymbol{u}_c(\boldsymbol{x}, t)$

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Note that this is a generic feature of Boltzmann equation, while integrating the distribution over all the momenta, we have taken the 0th moment of the Boltzmann equation, which depends on the velocity. The equation for the velocity can be obtained by taking the second first moment of the Boltzmann equation which is integrating over $\frac{d^3p}{(2\pi)^3} p \frac{\hat{p}^j}{E}$

In principle the equation we will derive will depends on the second moment, and so on, there is an infinite hierarchy of Boltzmann equation, in practice however we close the hierarchy using the fact that the higher moment become negligible, In the case of the DM, the key assumption is that p/E is small, that is the

velocity of dark matter particule is small: the dark matter is cold.

Going back to our Boltzmann equation

$$\frac{\partial f_c}{\partial t} + \frac{\partial f_c}{\partial x^i}\frac{p}{aE}\hat{p}^i - \left[(H+\dot{\phi})p + \frac{E}{a}\hat{p}^i\frac{\partial \psi}{\partial x^i}\right]\frac{\partial f_c}{\partial p} = 0$$

And taking its first moment

$$\begin{aligned}\frac{\partial}{\partial t}\int \frac{d^3p}{(2\pi)^3} p\frac{\hat{p}^j}{E} f_c \quad &+ \quad \frac{1}{a}\,\frac{\partial}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} f_c \frac{p^2}{E(p)^2}\hat{p}^i\hat{p}^j - (H+\dot{\phi})\int \frac{d^3p}{(2\pi)^3} p^2\frac{\hat{p}^j}{E}\frac{\partial f_c}{\partial p} \\ &- \quad \frac{1}{a}\frac{\partial \psi}{\partial x^i}\int \frac{d^3p}{(2\pi)^3} p\hat{p}^j\hat{p}^i\frac{\partial f_c}{\partial p} = 0\end{aligned}$$

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Second order in p/E

$$\frac{\partial}{\partial t}\int\frac{d^3p}{(2\pi)^3}p\frac{\hat{p}^j}{E}f_c + \cancel{\frac{1}{a}\frac{\partial}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}f_c\frac{p^2}{E(p)^2}}\hat{p}^i\hat{p}^j - (H+\dot{\phi})\int\frac{d^3p}{(2\pi)^3}p^2\frac{\hat{p}^j}{E}\frac{\partial f_c}{\partial p}$$
$$- \frac{1}{a}\frac{\partial\psi}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}p\hat{p}^j\hat{p}^i\frac{\partial f_c}{\partial p} = 0$$

And the two other terms can be integrated by part, finally we get

$$\frac{\partial(n_c u_c^j)}{\partial t} + 4Hn_c u_c^j + \frac{n_c}{a}\frac{\partial\psi}{\partial x^j} = 0$$

Which we can simplify using the equation on n_c

$$\frac{\partial n_c}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}(n_c u_c^i) + 3(H+\dot{\phi})n_c = 0$$

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Finally

$$\boxed{\frac{\partial u_c^j}{\partial t} + Hu_c^j + \frac{1}{a}\frac{\partial \psi}{\partial x^j} = 0}$$

Again this look like
a standard fluid equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\nabla p}{\rho} = \mathbf{g}$$

So we finally have our equations for the evolution of the dark matter, both at the background and at the perturbation level

$$\boxed{\begin{aligned} &\frac{d\bar{n}_c}{dt} + 3H\bar{n}_c = 0 \\ &\frac{\partial \delta_c}{\partial t} + \frac{1}{a}\,\frac{\partial u_c^i}{\partial x^i} + 3\dot{\phi} = 0 \\ &\frac{\partial u_c^j}{\partial t} + H u_c^j + \frac{1}{a}\frac{\partial \psi}{\partial x^j} = 0 \end{aligned}}$$

What makes the dark matter special is the fact that it's non interacting, if we were to redo the computation for the baryons, we would have to include interactions

$$\frac{\partial f_b}{\partial t} + \frac{\partial f_b}{\partial x^i}\frac{p}{aE}\hat{p}^i - \left[(H+\dot{\phi})p + \frac{E}{a}\hat{p}^i\frac{\partial \psi}{\partial x^i}\right]\frac{\partial f_b}{\partial p} = C[f_b, f_\gamma]$$

Baryons interact through Compton scattering and Coulomb scattering

$$e^- + \gamma \longleftrightarrow e^- + \gamma \qquad\qquad e^- + p \longleftrightarrow e^- + p$$

Note that none of these interactions create or destroy particules, we therefore expect that the equations describing the densities are unaffected, it is indeed the case

$$\frac{d\bar{n}_b}{dt} + 3H\bar{n}_c = 0$$

$$\frac{\partial \delta_b}{\partial t} + \frac{1}{a}\frac{\partial u_b^i}{\partial x^i} + 3\dot{\phi} = 0$$

However, baryons are going to exchange momenta with photons, so the Baryons velocity equation is modified

$$\frac{\partial u_b^j}{\partial t} + H u_b^j + \frac{1}{a}\frac{\partial \psi}{\partial x^j} = \boxed{\frac{4\rho_\gamma}{3\rho_b} n_e \sigma_T (u_\gamma^j - u_b^j)}$$

The baryons feel the radiation pressure of the photons ok, back to the full system of equations

$$
\begin{aligned}
\Theta' + ik\mu\,\Theta &= -\phi' - ik\mu\psi - \tau'\left[\Theta_0 - \Theta + \mu u_b - \frac{1}{2}P_2\mu\Pi\right] \\
\delta_c' + iku_c &= -3\phi' \\
u_c' + \frac{a'}{a}u_c &= -ik\,\psi \\
\delta_b' + iku_b &= -3\phi' \\
u_b' + \frac{a'}{a}u_b &= -ik\,\psi + \frac{\tau'}{R}[u_b + 3i\Theta_1]
\end{aligned}
$$

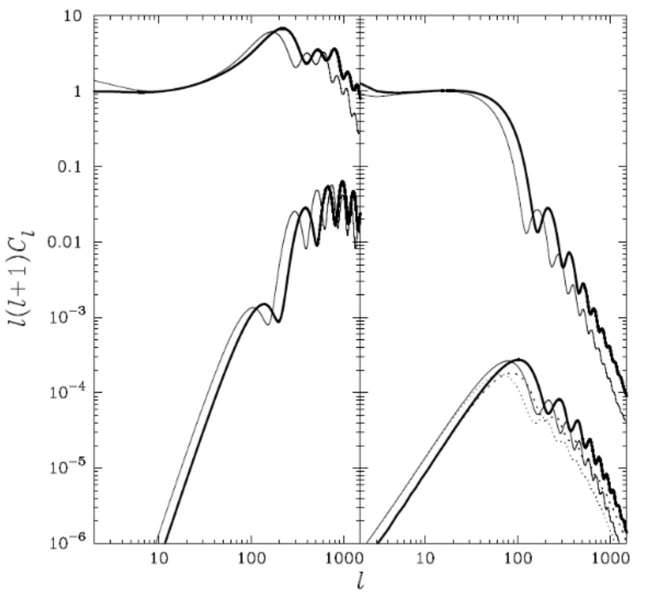
$$
\begin{aligned}
k^2\phi + 3\frac{a'}{a}\left(\phi' - \psi\frac{a'}{a}\right) &= 4\pi G a^2\left[\rho_c\delta_c + \rho_b\delta_b + \rho_\gamma\delta_\gamma + \rho_\nu\delta_\nu\right] \\
k^2(\phi + \psi) &= -32\pi G a^2\rho_r\Theta_{r,2} \sim 0
\end{aligned}
$$

In order to predict the CMB anisotropies we observe today

$$
\Theta\Big|_{\eta_R} = \int_0^{\eta_R} d\eta g(\eta)\left[(\Theta_0 + \psi) + \boldsymbol{e}\boldsymbol{u}_b\right] + 2\int_0^{\eta_R} d\eta \exp(-\tau)\dot{\psi}
$$

we need to solve all of these equations jointly

The standard in the community is to use public codes such as CAMB (fortran + python wrapper) or CLASS (C + python wrapper)



Code for Anisotropies in the Microwave Background

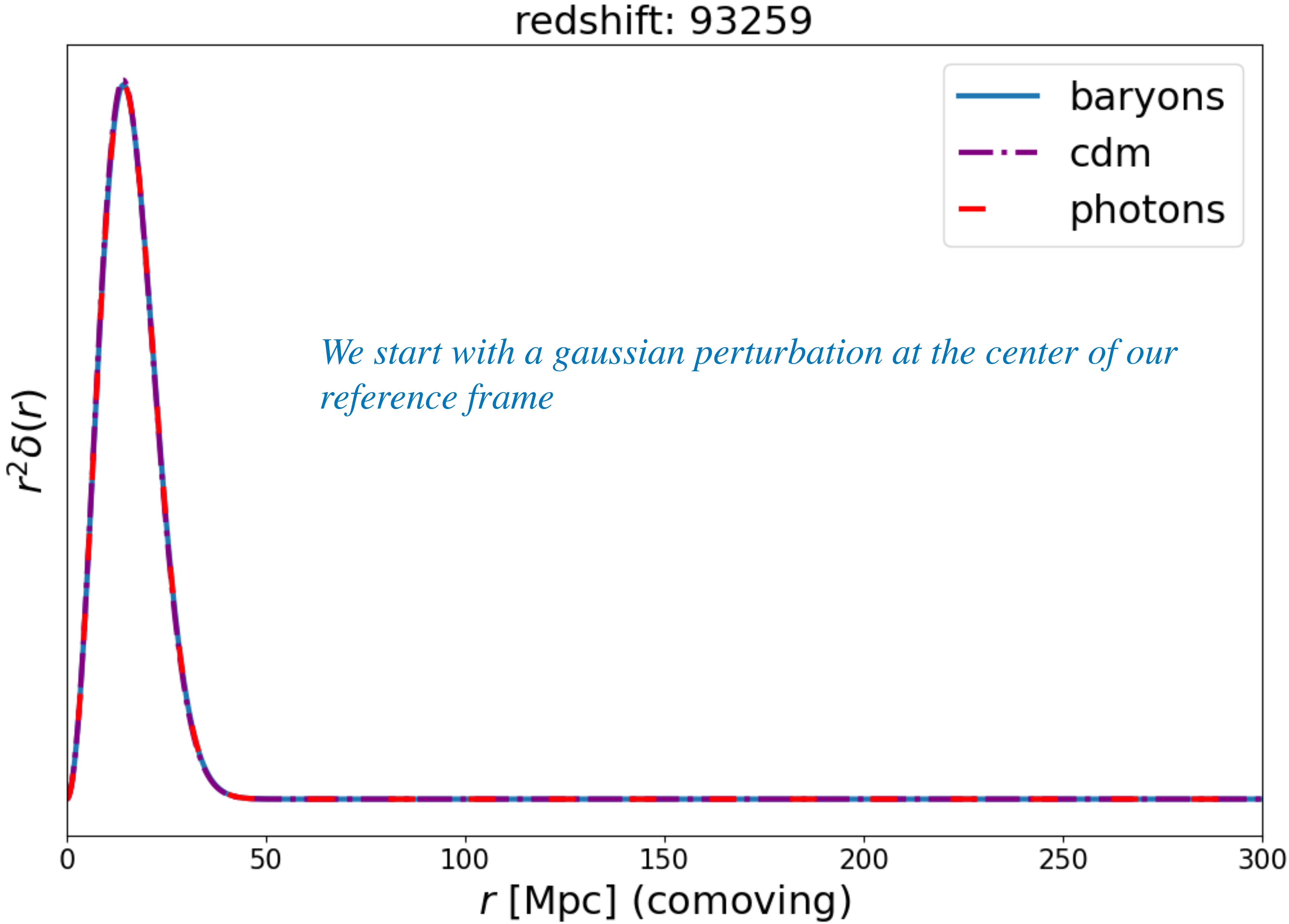
by Antony Lewis and Anthony Challinor

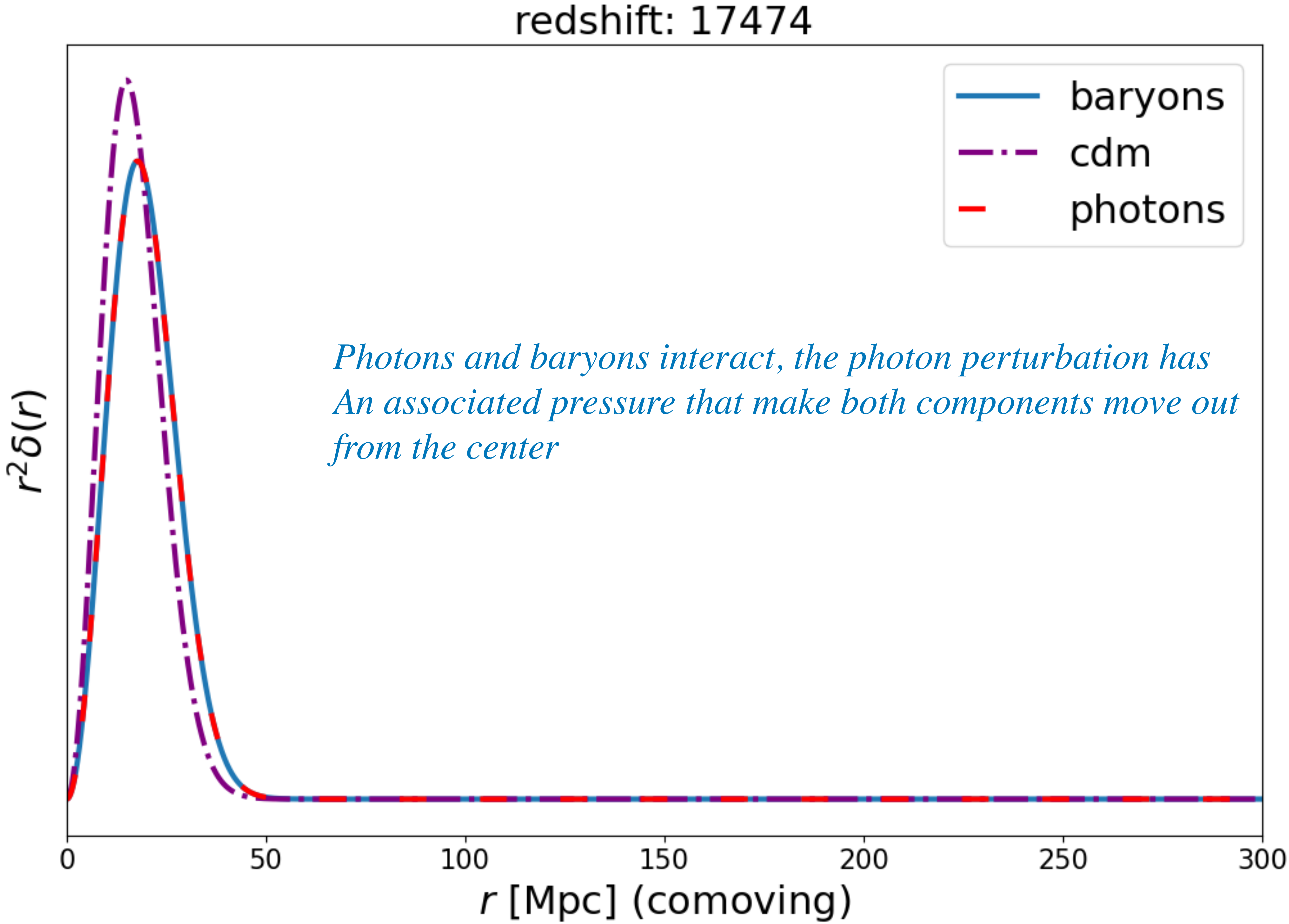
Get help: [ ] Search Google™ Custom Search

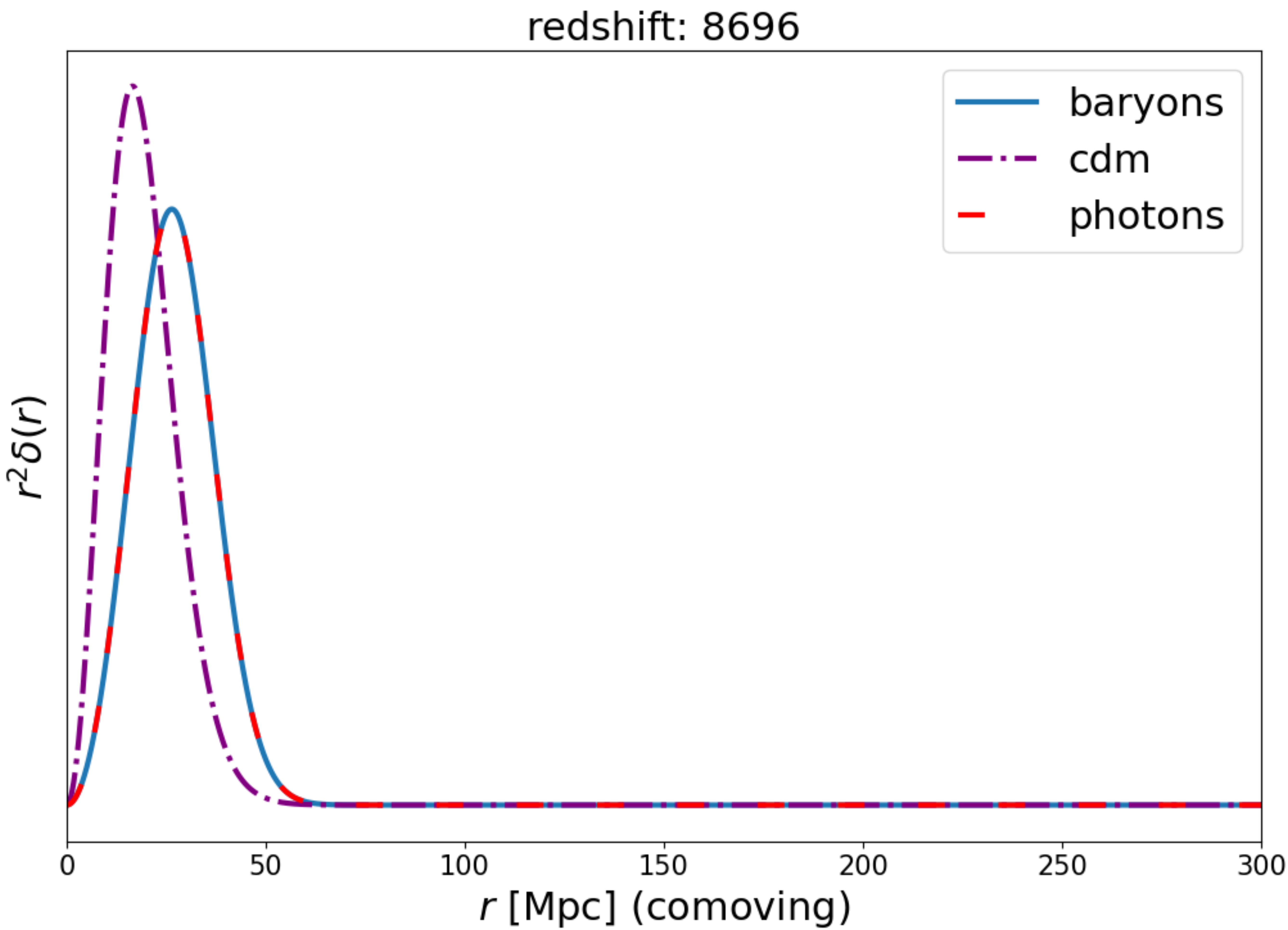
Features:

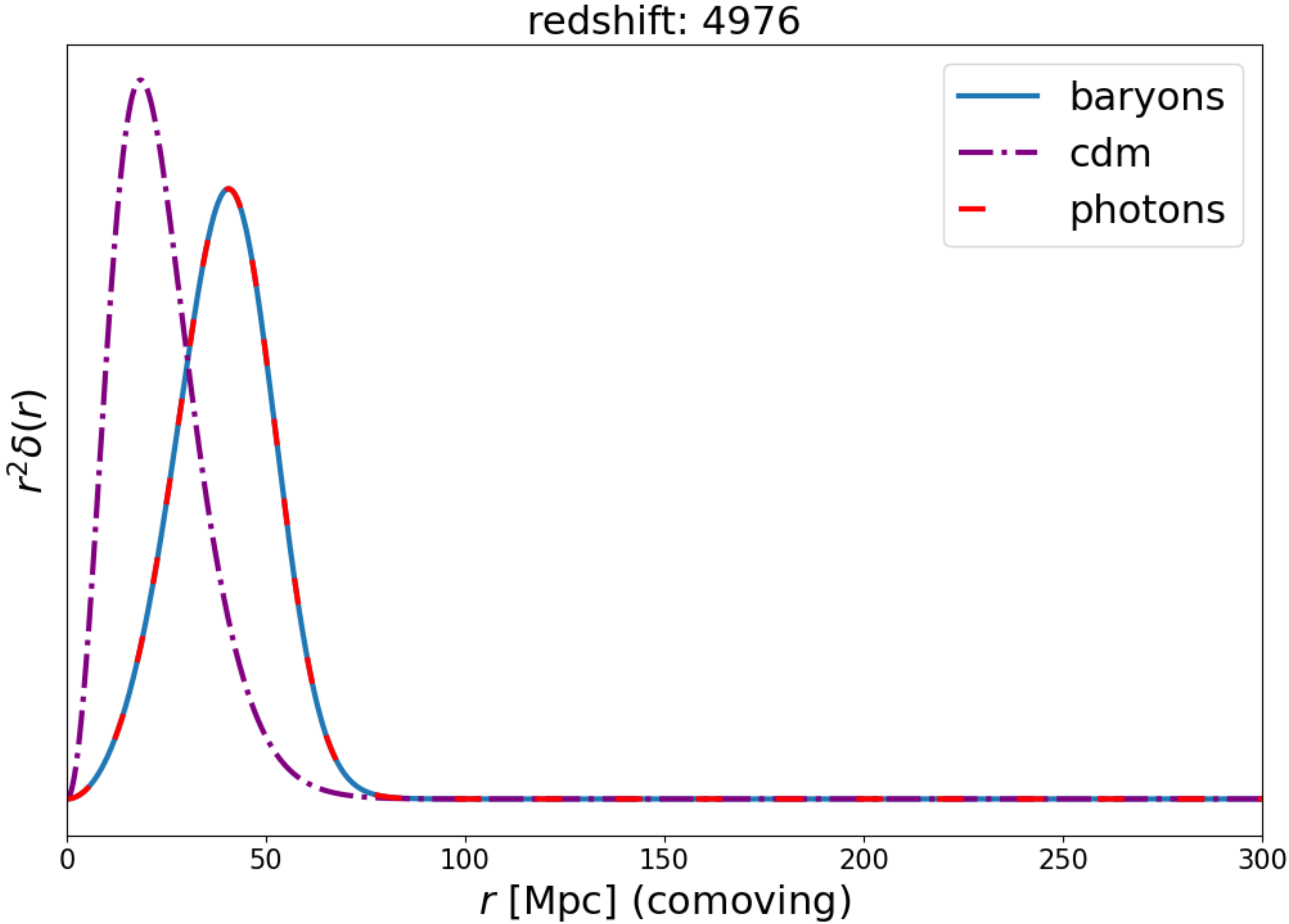
- Optimized Python and Fortran code
- Calculate CMB, lensing, source count and dark-age 21cm angular power spectra
- Matter transfer functions, power spectra, σ_8 and related quanties
- General background cosmology
- Support for closed, open and flat models
- Scalar, vector and tensor modes including polarization
- Fast computation to ~0.1% accuracy, with controllable accuracy level
- Object-oriented Python and easily-extensible modern Fortran 2008 classes
- Efficient support for massive neutrinos and arbitrary neutrino mass splittings
- Optional modelling of perturbed recombination and temperature perturbations
- Calculation of local primordial and CMB lensing bispectra (Fortran)

Let’s solve the equations in a very simple case, let’s assume that the Universe is composed of a very homogeneous background, with, as initial conditions,
a single perturbation in the middle, let’s assume that the perturbation contains
baryons, dark matter, and photons and let’s look how it will evolve according to our Boltzmann and Einstein equations

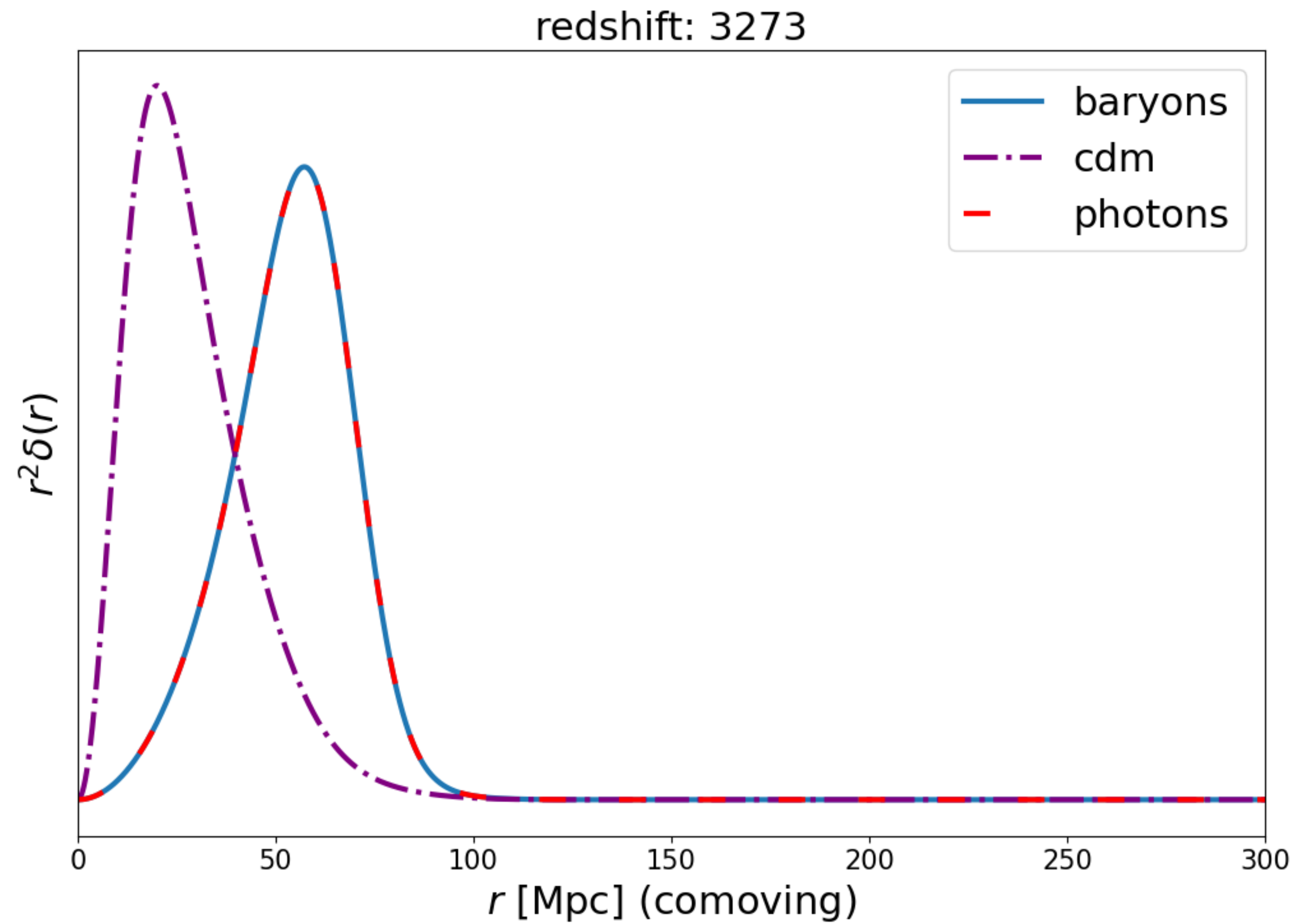

redshift: 93259
baryons
cdm
photons
We start with a gaussian perturbation at the center of our reference frame
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)

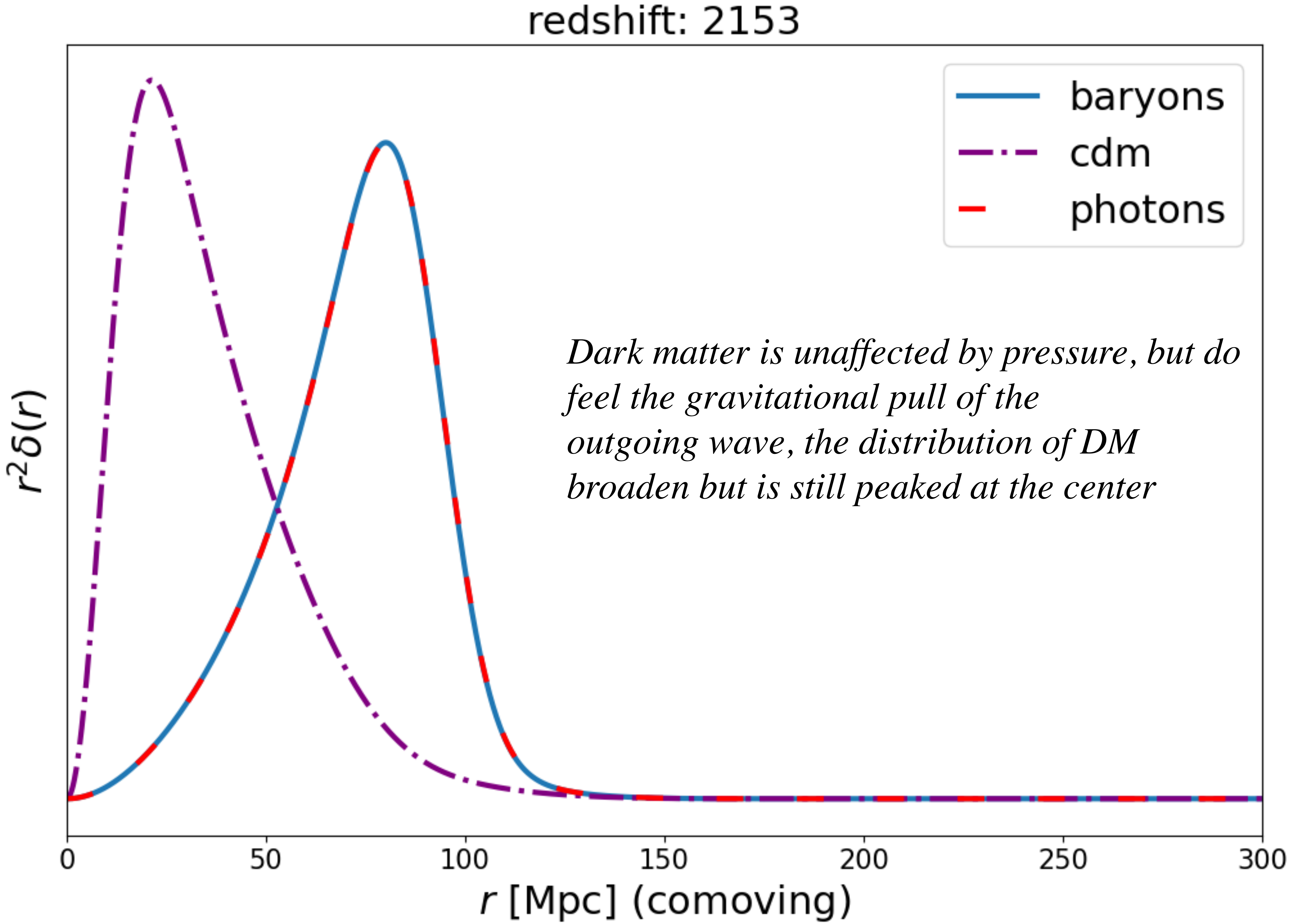

redshift: 17474
baryons
cdm
photons
Photons and baryons interact, the photon perturbation has
An associated pressure that make both components move out
from the center
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)

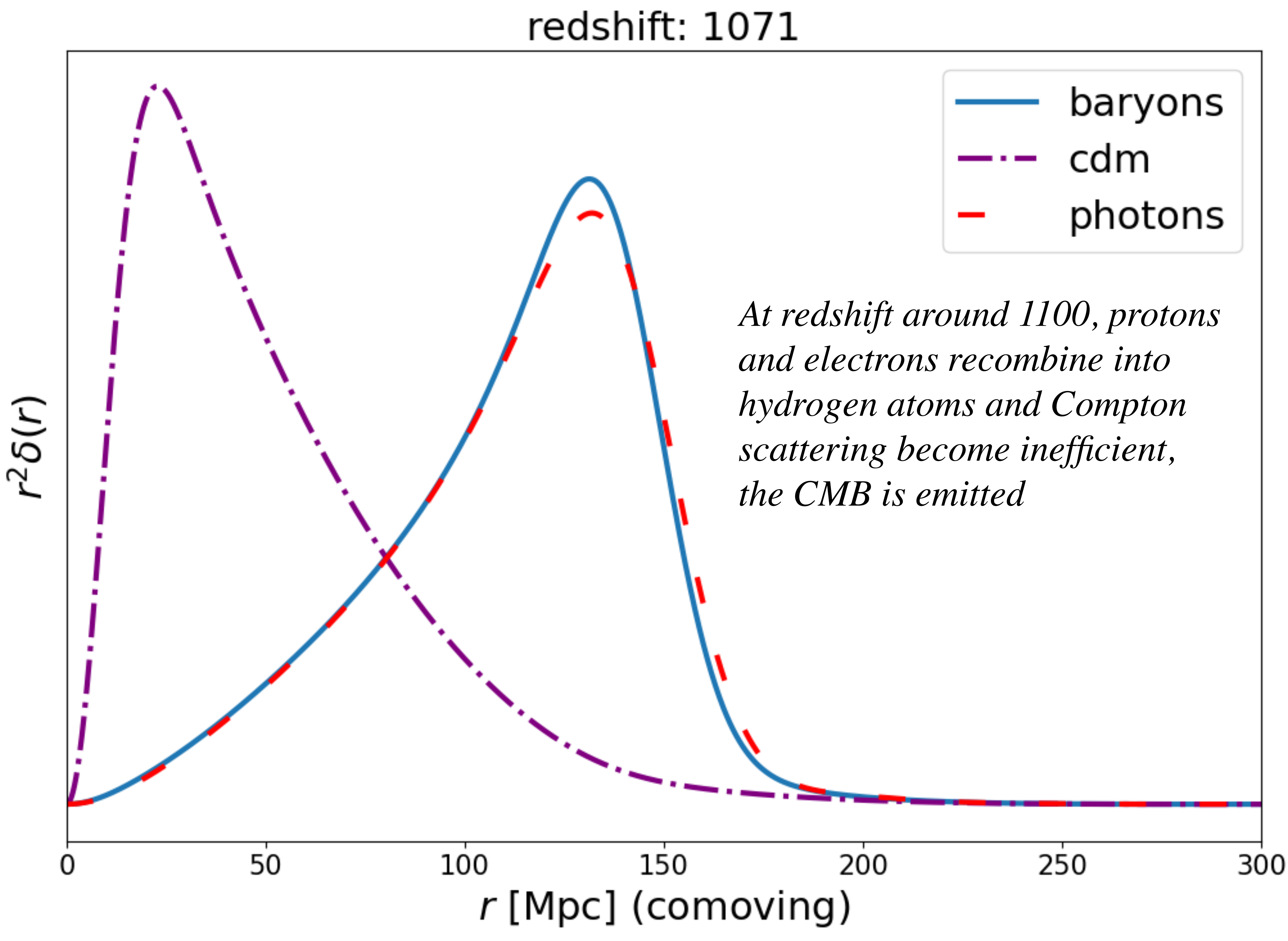

redshift: 8696
baryons
cdm
photons
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)

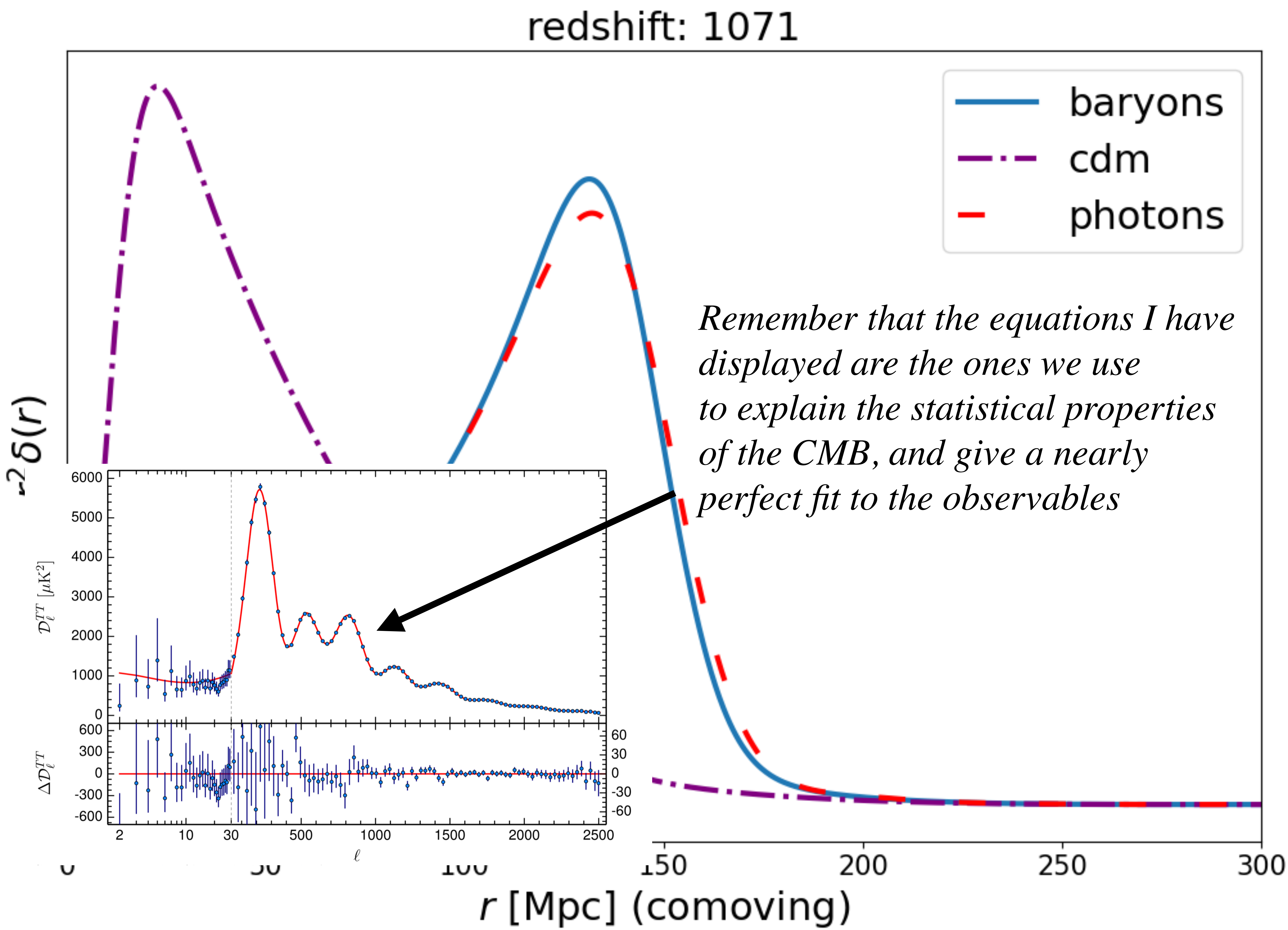


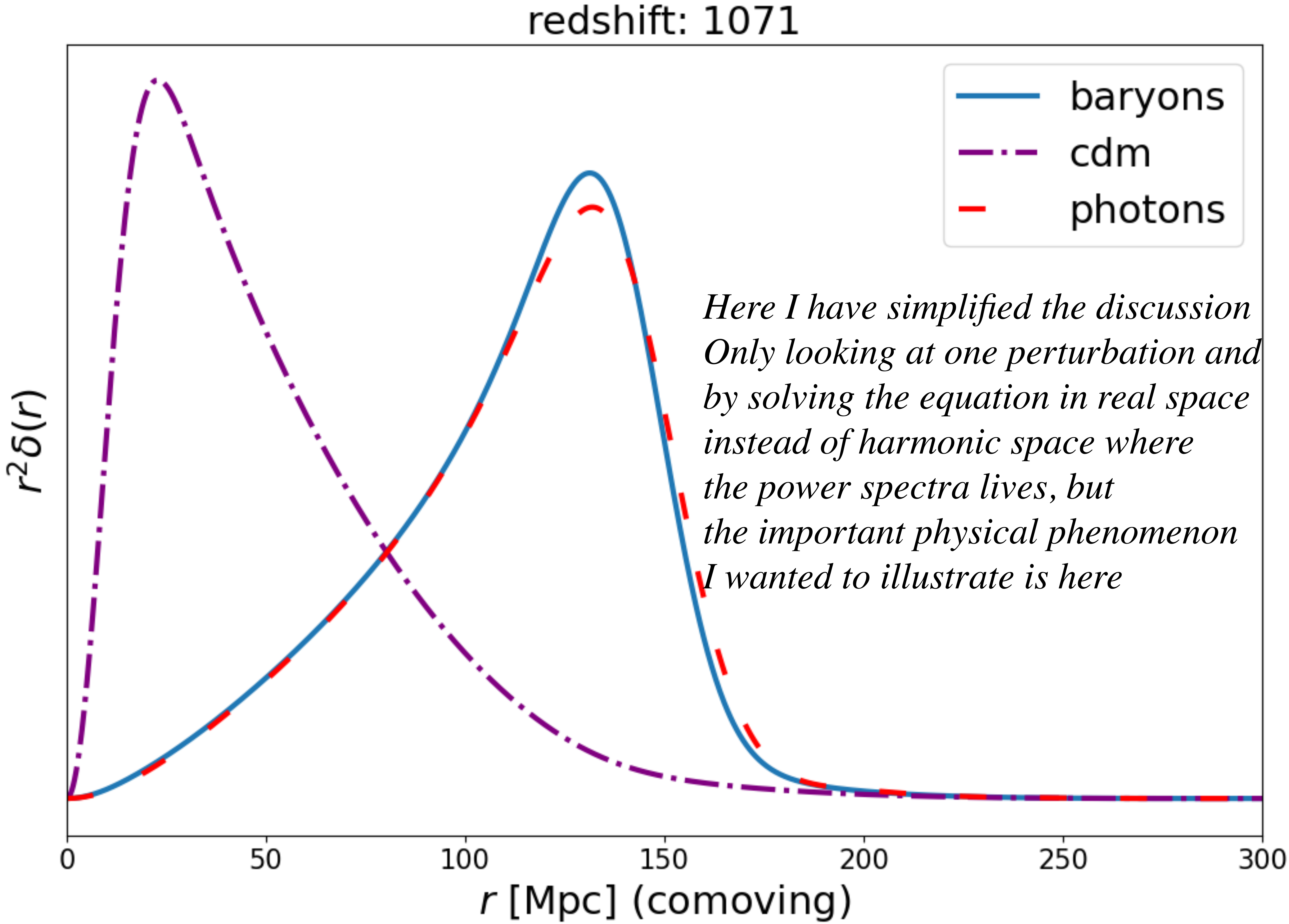
redshift: 4976
baryons
cdm
photons
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)

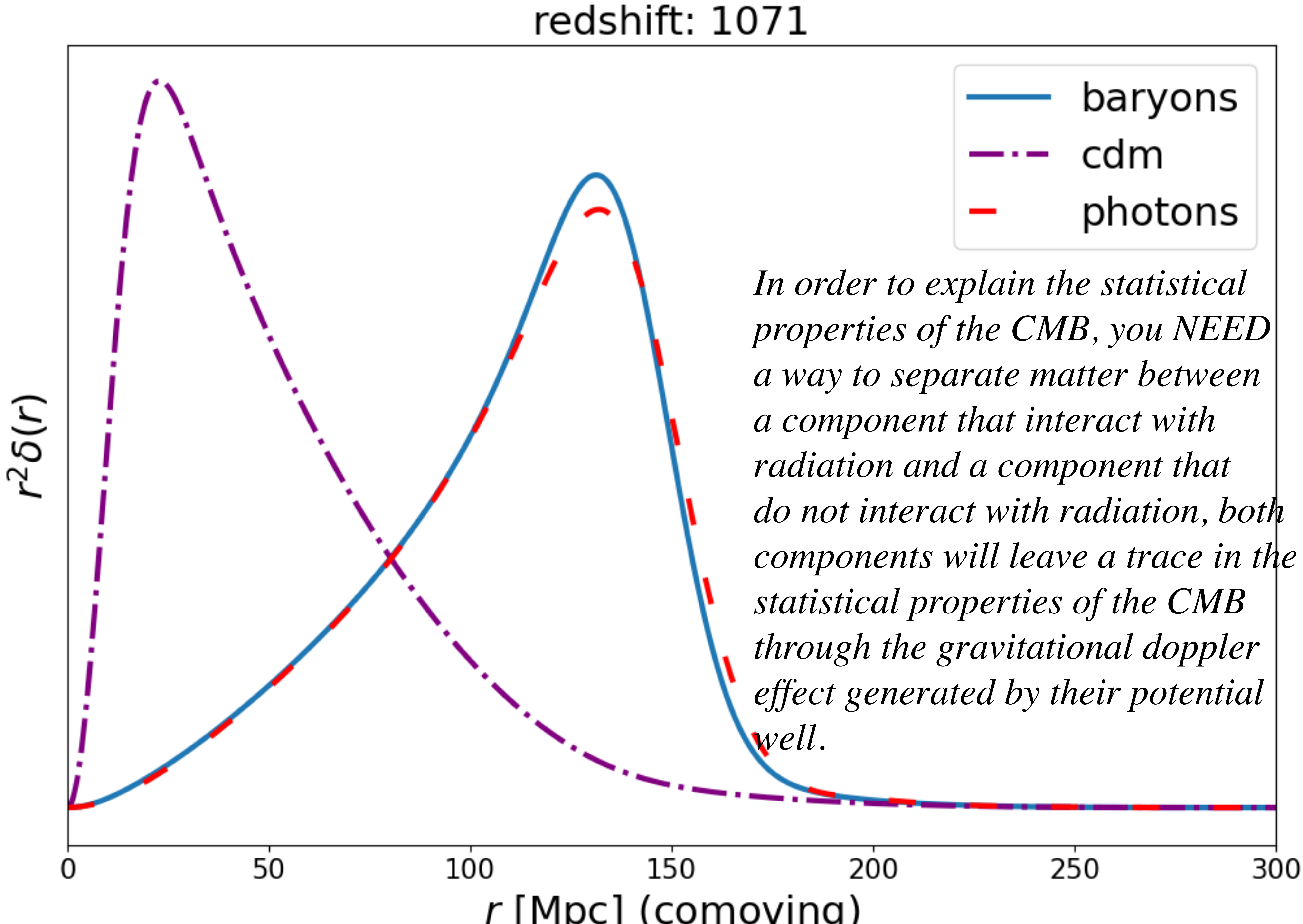

redshift: 3273
baryons
cdm
photons
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)


redshift: 2153
baryons
cdm
photons
Dark matter is unaffected by pressure, but do feel the gravitational pull of the outgoing wave, the distribution of DM broaden but is still peaked at the center
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)

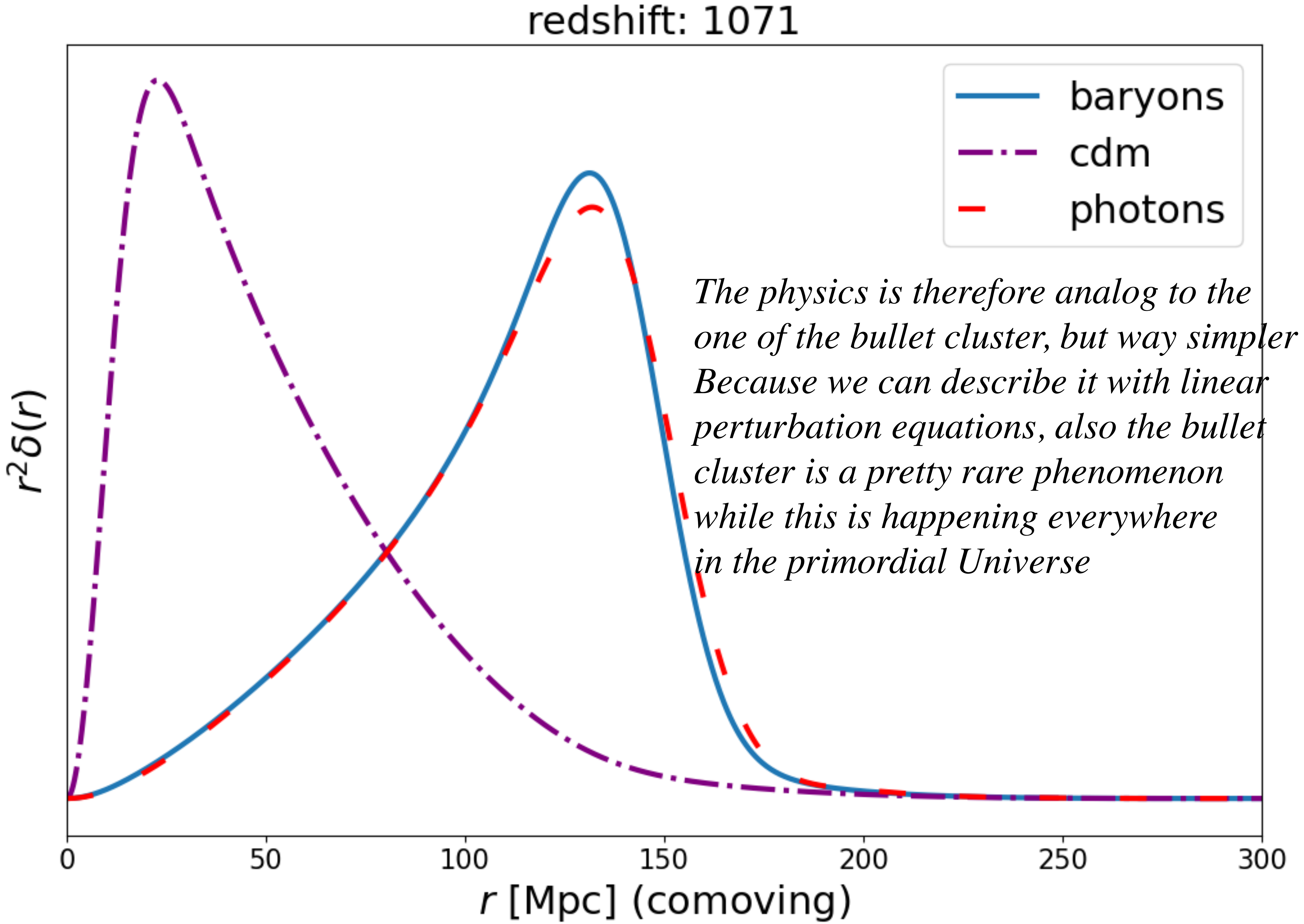

redshift: 1071
baryons
cdm
photons
At redshift around 1100, protons and electrons recombine into hydrogen atoms and Compton scattering become inefficient, the CMB is emitted
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)


redshift: 1071
baryons
cdm
photons
Remember that the equations I have displayed are the ones we use to explain the statistical properties of the CMB, and give a nearly perfect fit to the observables
$r^2\delta(r)$
r [Mpc] (comoving)
150
200
250
300
6000
5000
4000
3000
2000
1000
0
$\mathcal{D}_\ell^{TT}$ [μK^2]
600
300
0
-300
-600
$\Delta\mathcal{D}_\ell^{TT}$
60
30
0
-30
-60
2
10
30
500
1000
1500
2000
2500
ℓ

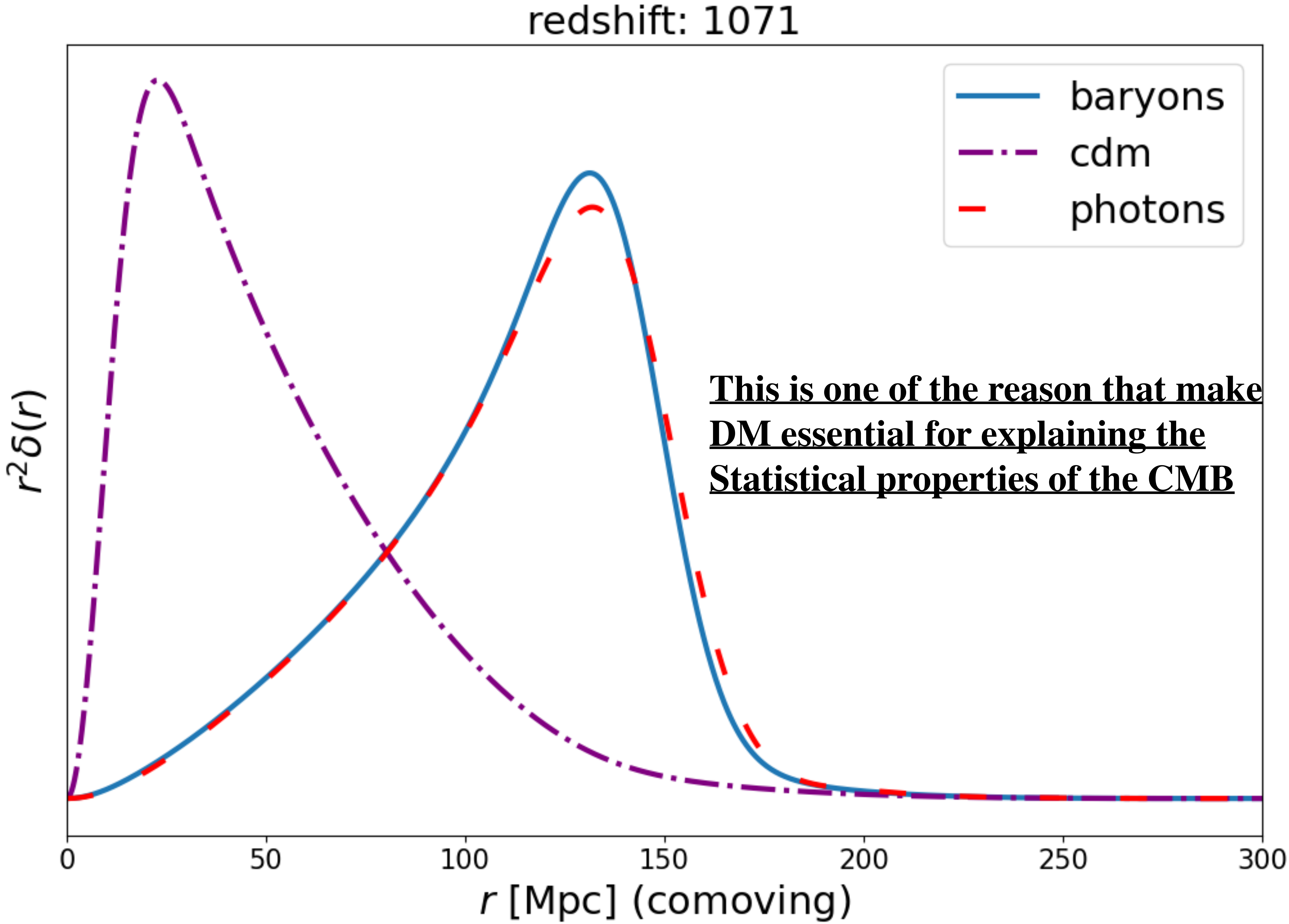

redshift: 1071
baryons
cdm
photons
Here I have simplified the discussion
Only looking at one perturbation and
by solving the equation in real space
instead of harmonic space where
the power spectra lives, but
the important physical phenomenon
I wanted to illustrate is here
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)

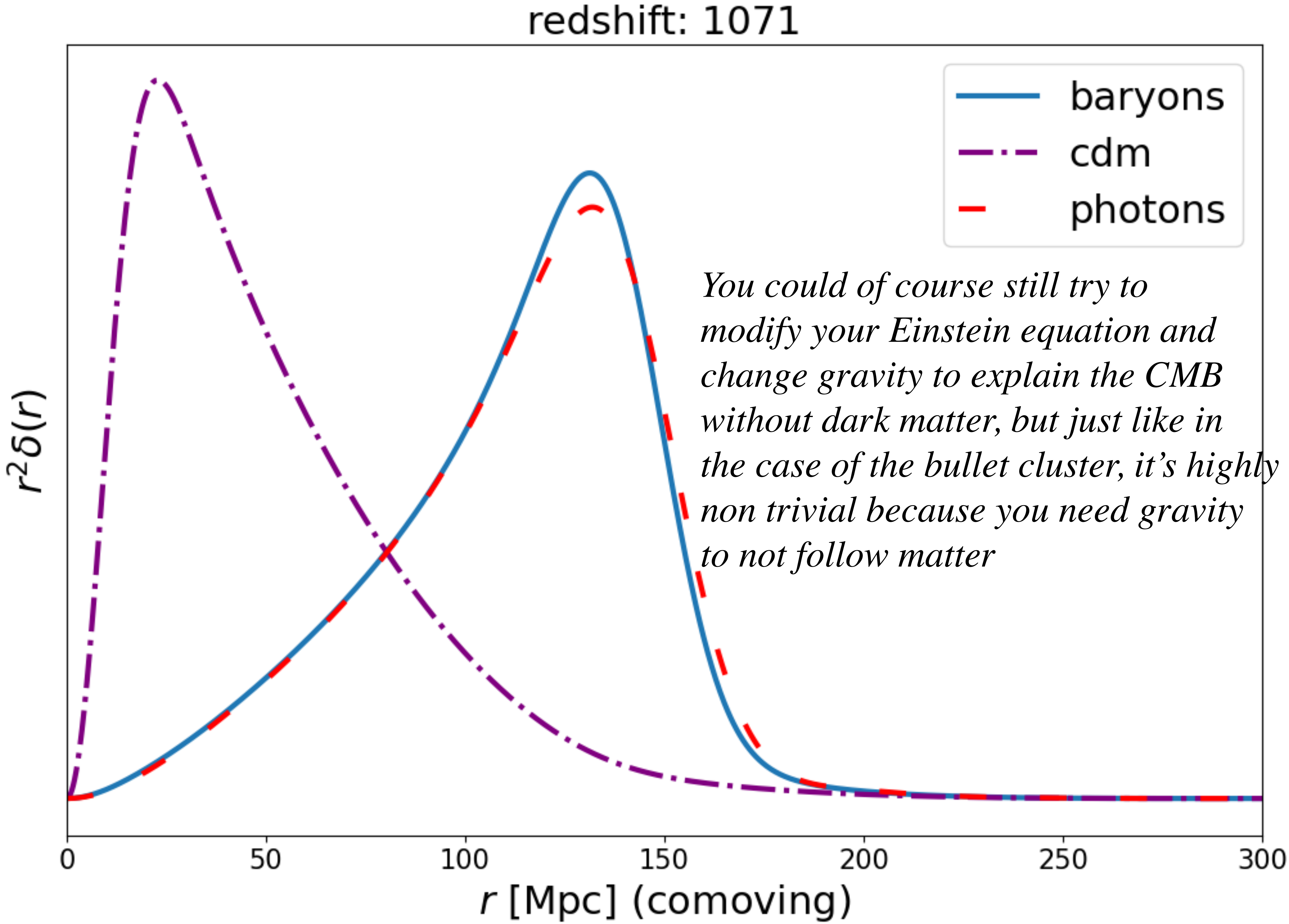


In order to explain the statistical properties of the CMB, you NEED a way to separate matter between a component that interact with radiation and a component that do not interact with radiation, both components will leave a trace in the statistical properties of the CMB through the gravitational doppler effect generated by their potential well.

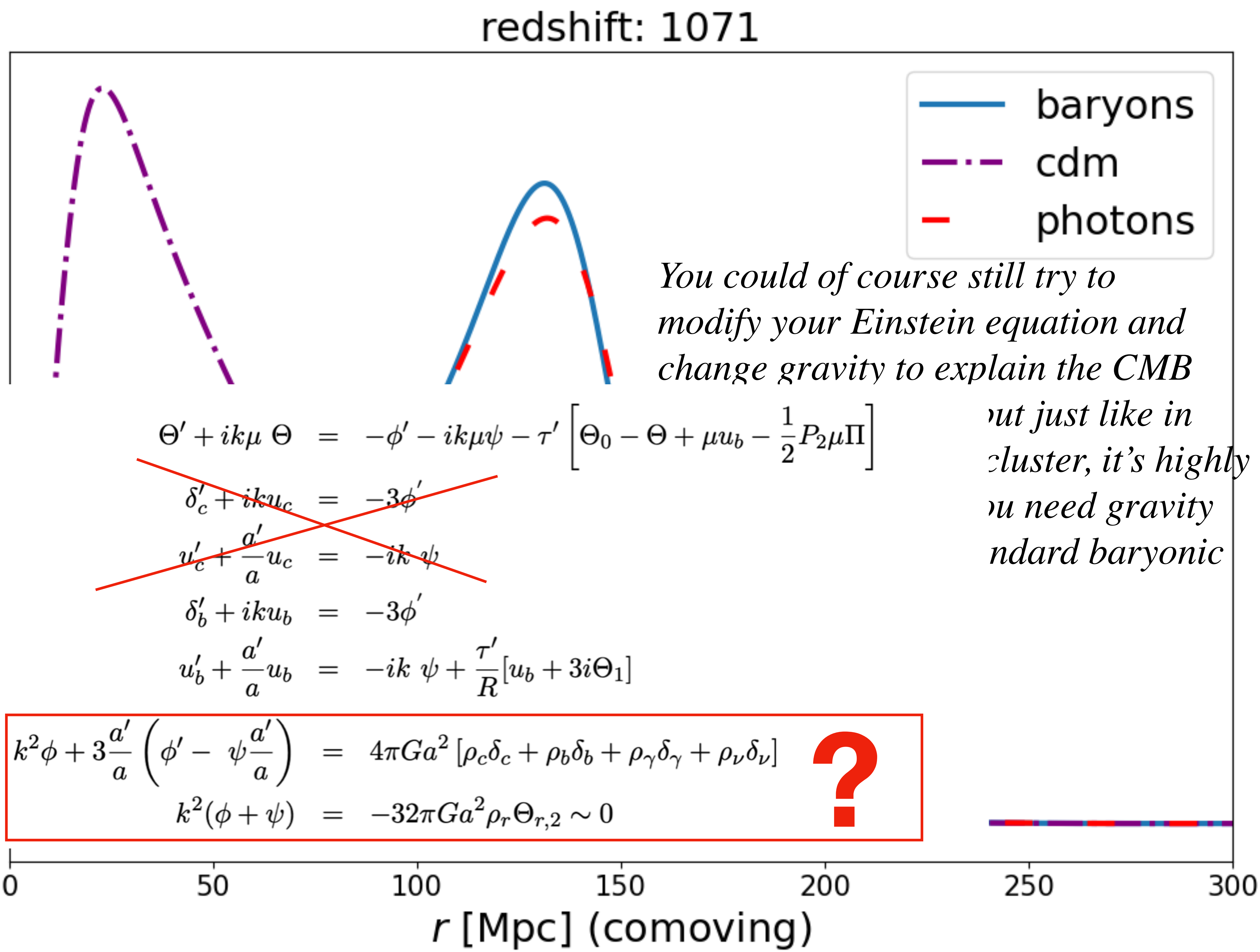


The physics is therefore analog to the one of the bullet cluster, but way simpler Because we can describe it with linear perturbation equations, also the bullet cluster is a pretty rare phenomenon while this is happening everywhere in the primordial Universe


redshift: 1071
baryons
cdm
photons
This is one of the reason that make DM essential for explaining the Statistical properties of the CMB
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)



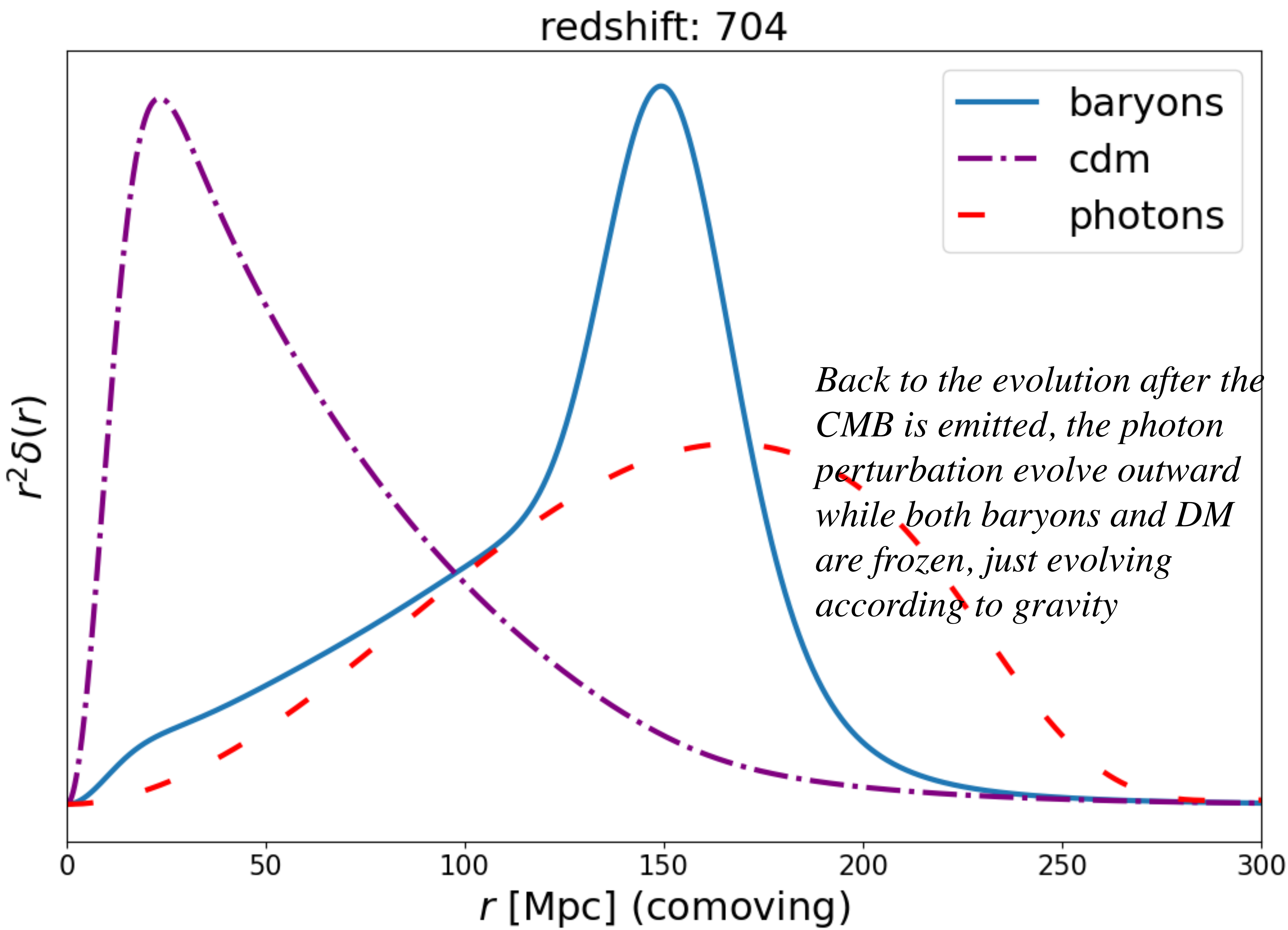
You could of course still try to modify your Einstein equation and change gravity to explain the CMB without dark matter, but just like in the case of the bullet cluster, it's highly non trivial because you need gravity to not follow matter



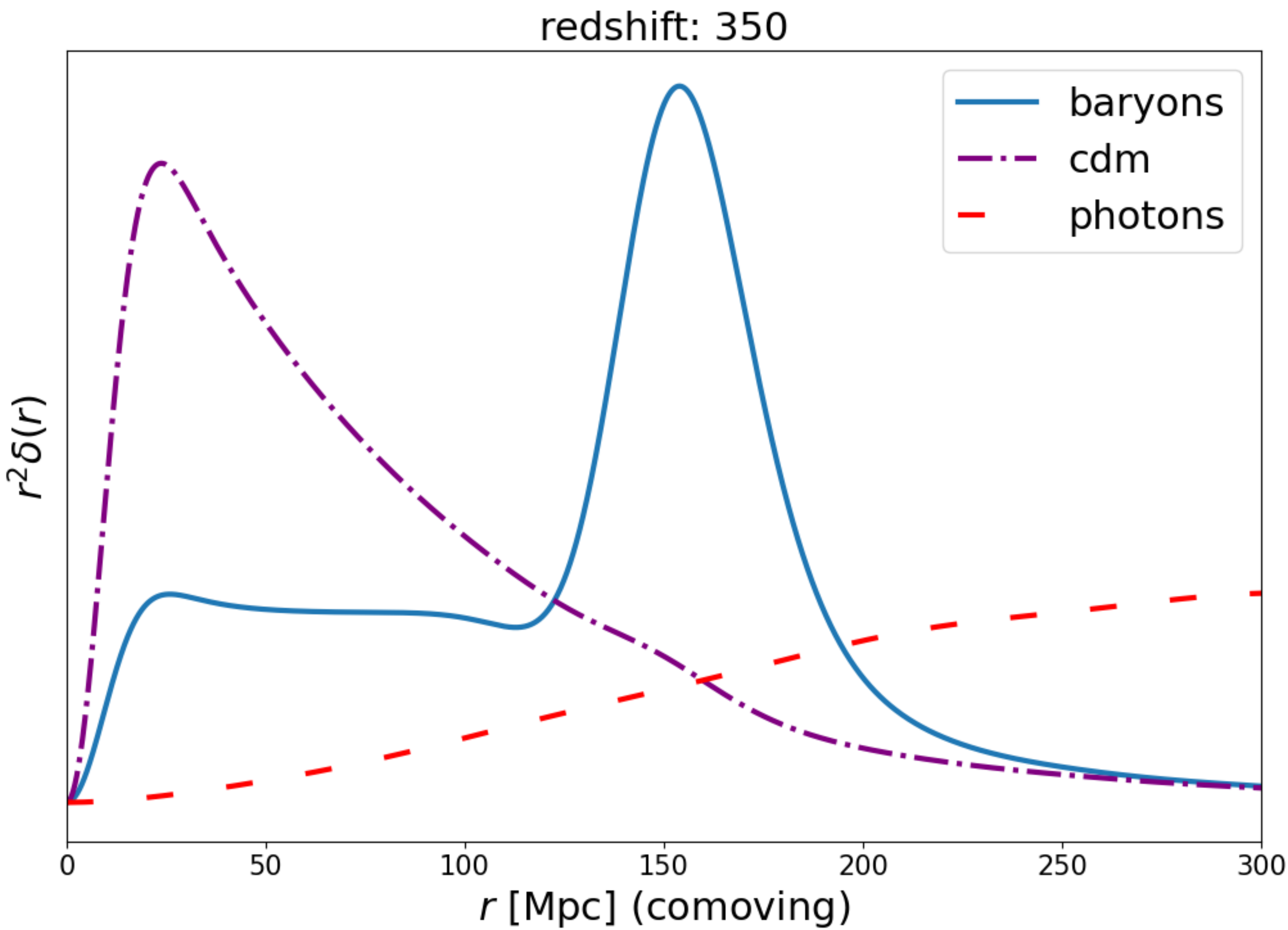
You could of course still try to modify your Einstein equation and change gravity to explain the CMB
…ut just like in
…cluster, it's highly
…u need gravity
…ndard baryonic

$$
\begin{aligned}
\Theta' + ik\mu\,\Theta &= -\phi' - ik\mu\psi - \tau'\left[\Theta_0 - \Theta + \mu u_b - \frac{1}{2}P_2\mu\Pi\right] \\
\cancel{\delta_c' + iku_c} &\cancel{= -3\phi'} \\
\cancel{u_c' + \frac{a'}{a}u_c} &\cancel{= -ik\,\psi} \\
\delta_b' + iku_b &= -3\phi' \\
u_b' + \frac{a'}{a}u_b &= -ik\,\psi + \frac{\tau'}{R}[u_b + 3i\Theta_1]
\end{aligned}
$$

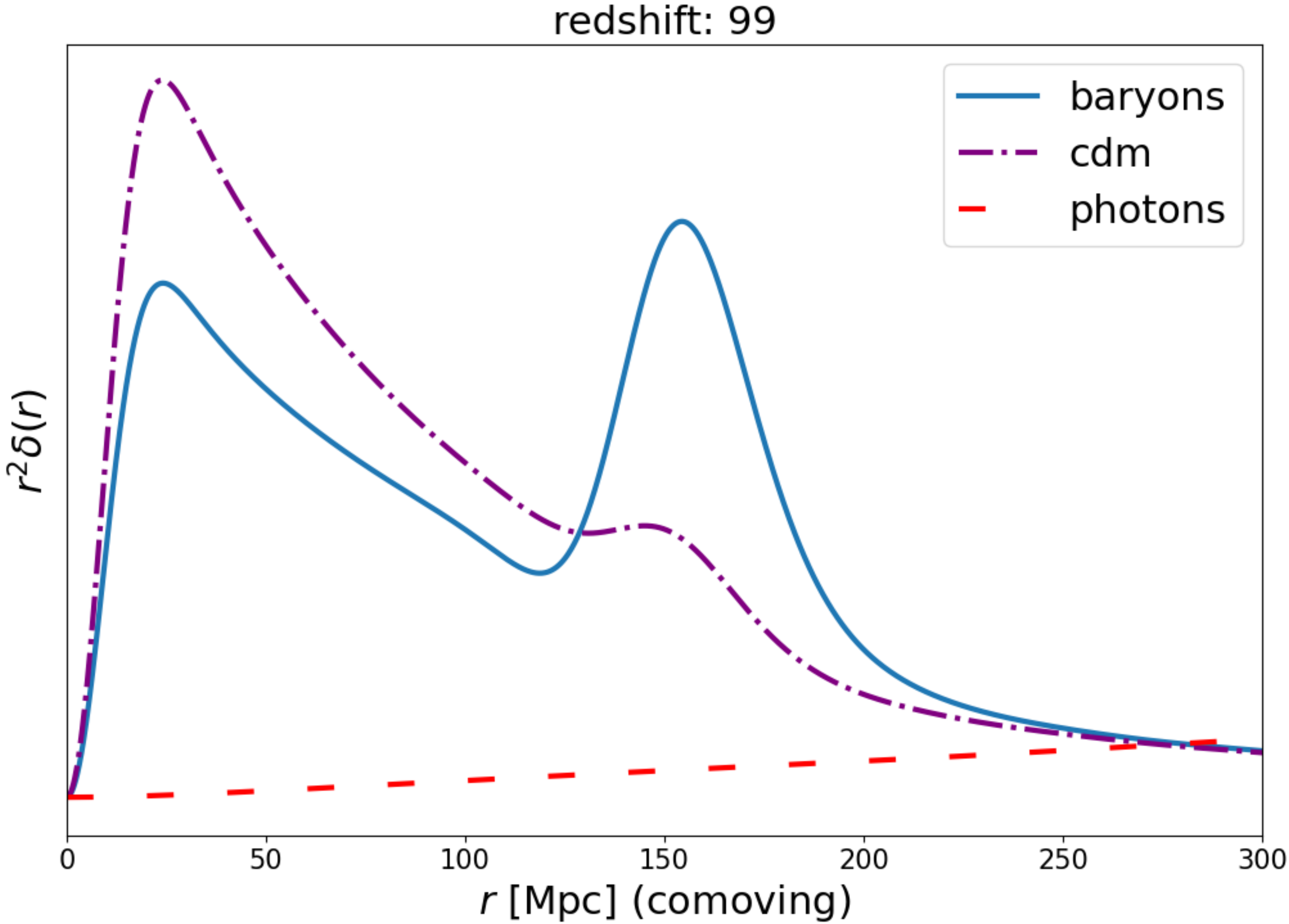
$$
\boxed{
\begin{aligned}
k^2\phi + 3\frac{a'}{a}\left(\phi' - \psi\frac{a'}{a}\right) &= 4\pi G a^2\left[\rho_c\delta_c + \rho_b\delta_b + \rho_\gamma\delta_\gamma + \rho_\nu\delta_\nu\right] \\
k^2(\phi + \psi) &= -32\pi G a^2\rho_r\Theta_{r,2} \sim 0
\end{aligned}
\quad ?}
$$



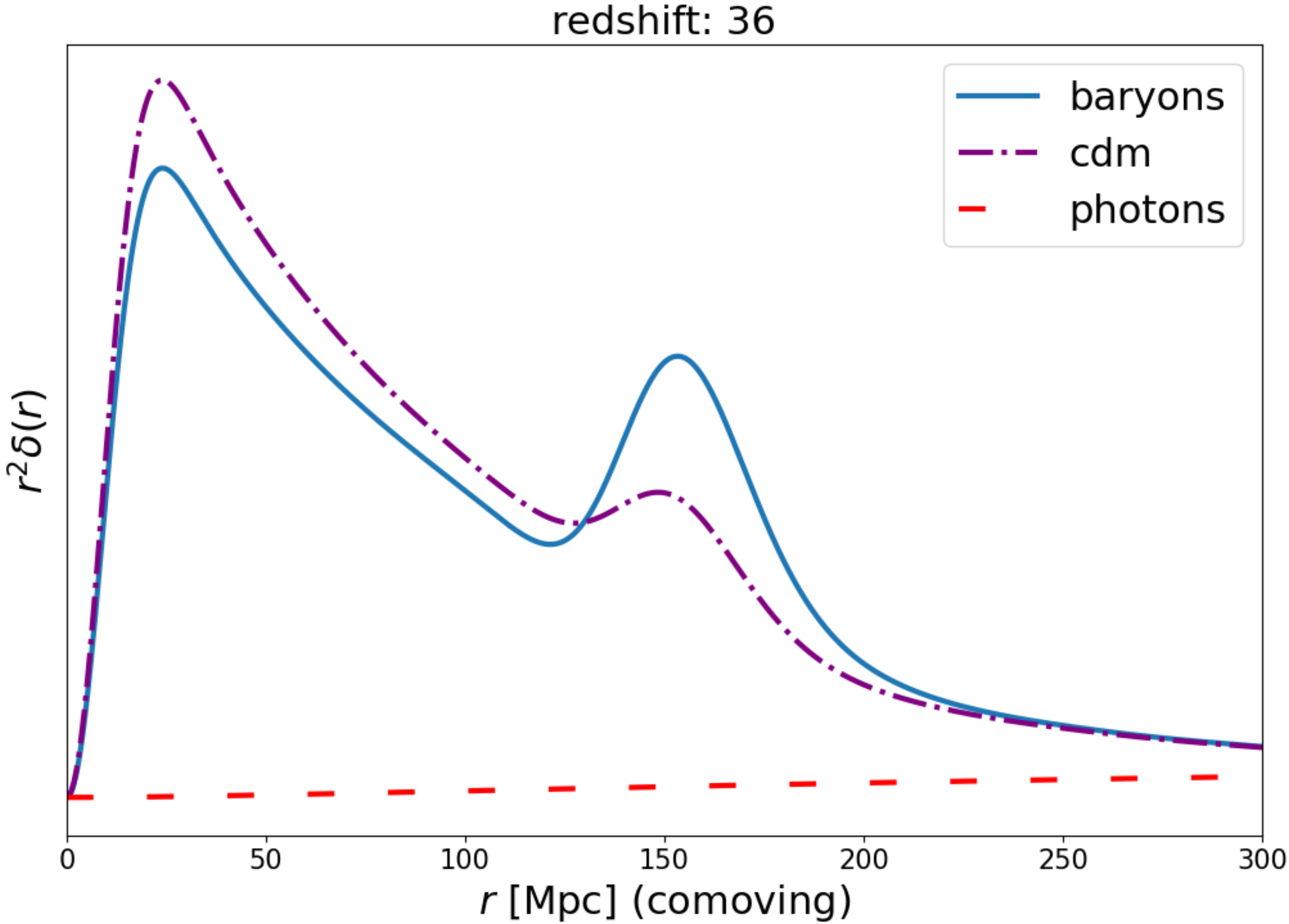
Back to the evolution after the CMB is emitted, the photon perturbation evolve outward while both baryons and DM are frozen, just evolving according to gravity



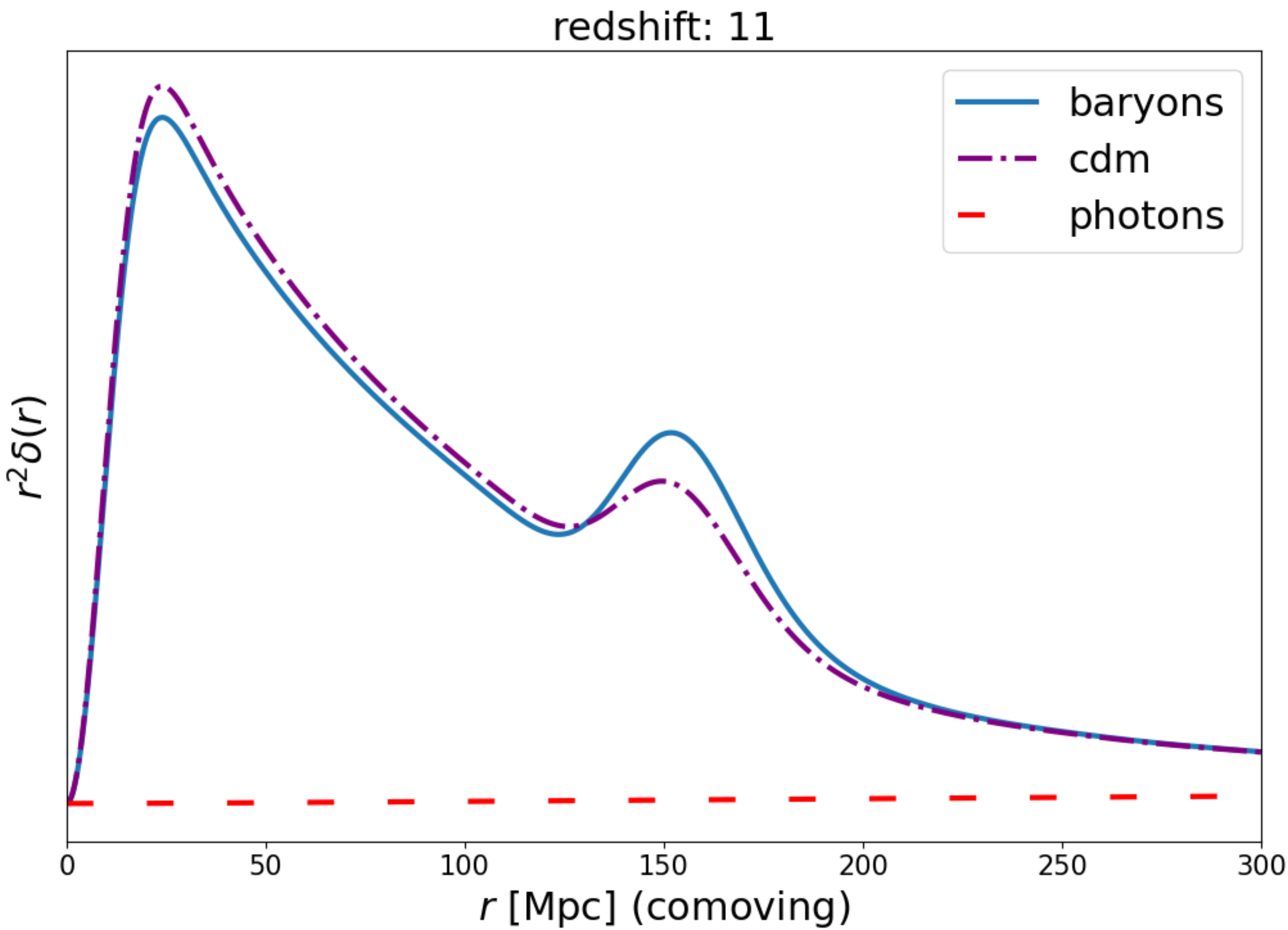
redshift: 350
baryons
cdm
photons
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)

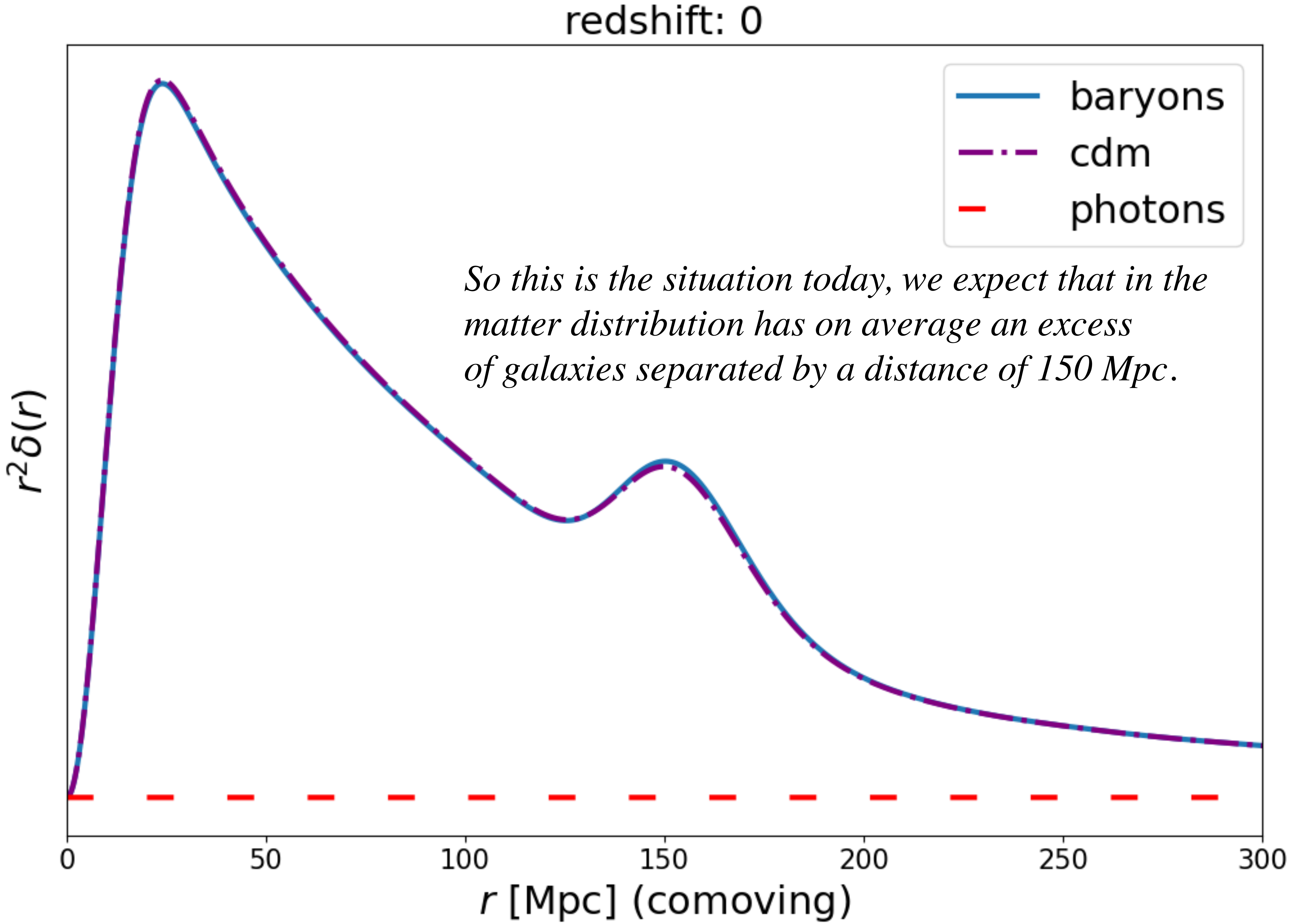


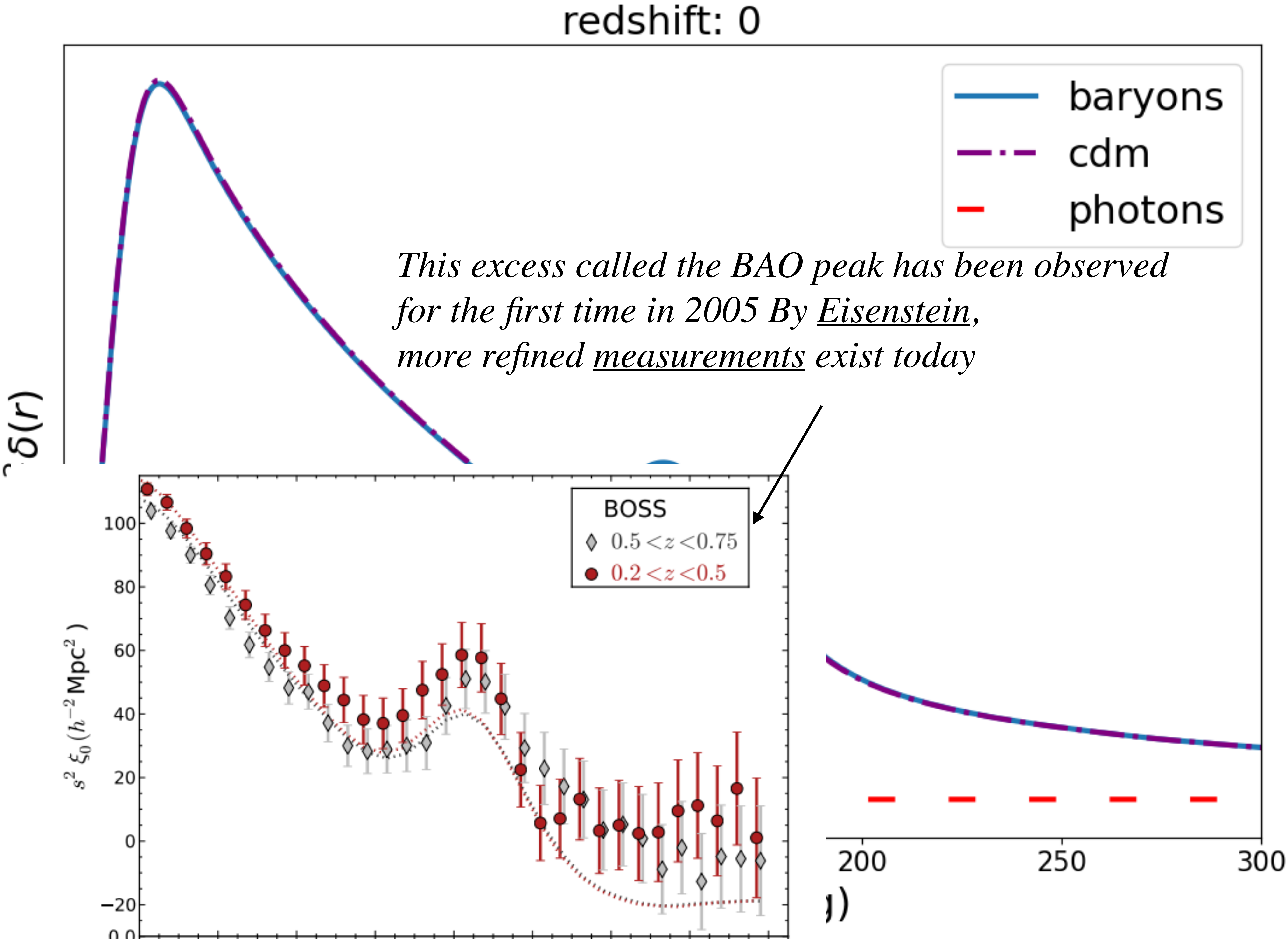
redshift: 99
baryons
cdm
photons
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)



redshift: 36
baryons
cdm
photons
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)


redshift: 11
baryons
cdm
photons
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)


redshift: 0
baryons
cdm
photons
So this is the situation today, we expect that in the matter distribution has on average an excess of galaxies separated by a distance of 150 Mpc.
$r^2\delta(r)$
0
50
100
150
200
250
300
r [Mpc] (comoving)



This excess called the BAO peak has been observed for the first time in 2005 By Eisenstein, more refined measurements exist today

See you next week !

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2)$$

$$d\eta = dt/a$$

Eta conformal time while t is the cosmological time

In reality, the decoupling is not instantaneous, so we have to compute the probability that a photon last scatter of an electron at a given time t

The probability for a photon to scatter on an electron per unit of time is given by

$$\frac{dP_{\text{scatter}}}{dt} = n_{\text{e}} c \sigma_T$$

Free electron density Thomson scattering cross section

We are interested by the probability that the photon scatters between t and t+dt, and Then propagate freely (last scatter), it is called the visibility function g(t)

$$g(t)dt = n_e c \sigma_T \ dt \times P_{\text{noscatter}}(t, t_0)$$

To compute $P_{\text{noscatter}}(t, t_0)$ we divide the time between t and t_0 in a set of n intervals $\delta_t = \frac{(t_0 - t)}{n}$

$$\begin{aligned} P_{\text{noscatter}}(t, t_0) &= \prod_{i=1}^{n} (1 - dP_{\text{scatter}}(t_i)\delta t) \sim \prod_{i=1}^{n} e^{-dP_{\text{scatter}}(t_i)\delta t} \sim e^{-\sum_{i=1}^{n} dP_{\text{scatter}}(t_i)\delta t} \\ &\overset{n \to \infty}{=} \exp\left(- \int_t^{t_0} n_e c \sigma_T \ dt\right) = \exp(-\tau) \end{aligned}$$

The visibility function as a function of redshift is given by

$$g(z) \equiv \frac{n_e c \sigma_{\rm T}}{(1+z)H(z)} e^{-\tau}$$